A Simple Argument Against
Cantor's Diagonal Procedure
(Which Purportedly Proves That the Set of Real Numbers is Uncountable)

by

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1. Cantor's Diagonal Procedure

(I have adapted the following bit from <http://www.scidiv.bcc.ctc.edu/Math/diag.htm> given in the Mathematics Web pages of the Belleview Community College, Washington State, USA).

Cantor argues as follows: Suppose that the infinity of real numbers *is* countable — *i.e.,* supposing, say, the set of decimal numbers between zero and one is the *same* as the infinity of counting numbers. Then the decimal numbers *can* be put in one-to-one correspondence with the decimal numbers in a list, and the list can be put in a table, thus:

Table 1

Natural numbers	Decimal Numbers Between 0 and 1
1	$D_1 = 0.d_{11}d_{12}d_{13}d_{14} \cdots d_{1k} \cdots$
2	$D_2 = 0.d_{21}d_{22}d_{23}d_{24} \cdots d_{2k} \cdots$
3	$D_3 = 0.d_{31}d_{32}d_{33}d_{34} \cdots d_{3k} \cdots$
4	$D_4 = 0.d_{41}d_{42}d_{43}d_{44} \cdots d_{4k} \cdots$
…	
n	$D_n = 0.d_{n1}d_{n2}d_{n3}d_{n4} \cdots d_{nk} \cdots$
(etc)	

Here, **D** represents any decimal number between **0** and **1**, **d** represents any digit between **0** and **9** inclusive, the first subscript of any particular **d** is the natural number to which that particular **d** corresponds, and the second subscript of that same **d** is the number of places that particular **d** lies to the right of the decimal point.

Now each row of the table represents, of course, a natural number put in one-to-one correspondence with a decimal number between **0** and **1**.

Now consider the decimal number $\mathbf{X = 0.x_1x_2x_3x_4x_5 \ldots x_k} \ldots$, where $\mathbf{x_1}$ is any digit other than $\mathbf{d_{11}}$; $\mathbf{x_2}$ is different from $\mathbf{d_{22}}$; $\mathbf{x_3}$ is not equal to $\mathbf{d_{33}}$; $\mathbf{x_4}$ is not $\mathbf{d_{44}}$; and so on. Now, **X** *is* a decimal number, and **X** *is* between **0** and **1**, so it *should* be in our list. *But where is it?*

The decimal number **X** can't be the *first* in the list*,* since the first digit of **X** differs from the first digit of $\mathbf{D_1}$. Similarly, **X** can't be *second* in the list, because **X** and $\mathbf{D_2}$ have different hundredths-place digits; and **X** can't be *third* in the list, because **X** and $\mathbf{D_3}$ have different thousandth-place digits. In general, **X** can't be the $\mathbf{n}^{\text{th}}$ in the list — *i.e.,* it cannot be equal to $\mathbf{D_n}$ — since their $\mathbf{n}^{\text{th}}$ digits are not the same.

The decimal **X** is *nowhere* to be found in the list, no matter how large **n** gets. In other words, we have exhibited a decimal number that ought to be present in a huge humongous giganto list —0 but it isn't. No matter how we try to list the decimal numbers, and how long the list gets, at least *one* decimal number will be left out.

Therefore, argues Cantor, putting the decimal numbers in one-to-one correspondence with the natural numbers is impossible; and so the infinity of decimal numbers must be greater than the infinity of counting numbers.

... Q.E.D. (Or so says Cantor).

2. Simple Argument *Against* Cantor's Diagonal Procedure

But we can easily counter Cantor's argument. *Yes,* we can say, it *is* true that the decimal number $\mathbf{X = 0.x_1x_2x_3x_4x_5 \ldots x_k} \ldots$ is not in the part of the list *above* the natural number **n**. But there is an infinitely huge part of the list of natural numbers *after* **n**! This infinite list is compressed and almost hidden by those three small letters *"**etc.**"* at the bottom of Table 1. They may be small, but those three small letters *"**etc.**"* represent a list of naturals *infinitely* longer than the list *above* **n**.

All that Cantor *has* proved is that **X** is not on the list *above* **n**, no matter how large **n** may be. But he has *not* proved that **X** is not in the (not-yet-listed — and indeed *unlistable,* because infinitely large) part of the table *after* **n** … and he *cannot* prove that! After all, no matter how large **n** gets, the part of the table *after* **n** still remains infinitely larger than the part of the table *before* **n**.

And note that no matter how large **n** gets, the list *after* **n** can never be "brought up" to be included in the part of the list *before* **n**. This is because the part of the list *before* **n** is *necessarily* finite, while the part of the list *after* **n** must be infinitely long!

(That is, of course, if you really accept that the natural numbers are infinite — because if, like the Intuitionists, Constructivists and Finitists, you don't accept that, this whole argument is moot.)

In any case, we have demonstrated above that Cantor has *not* proved his thesis, *viz.,* that all the decimals between **0** and **1** cannot be put into one-to-one correspondence with the natural numbers. His entire argument is based on tricking you into believing that the list is complete, when in actual fact it isn't … and never can be.

3. Argument *Disproving* Cantor's Diagonal Procedure

Now we admit that in Section 2 above we have not actually *disproved* Cantor's argument. But we *have* proved that Cantor has not proved his *own* thesis, namely that all the decimals between **0** and **1** *cannot* be put into one-to-one correspondence with the natural numbers.

In this Section we shall actually *disprove* Cantor's thesis: we shall show that the decimal number **X** = $\mathbf{0.x_1x_2x_3x_4x_5} \ldots \mathbf{x_k} \ldots$ *must* be in Table 1 — specifically, on the list *after* **n**.

To do this we calculate the probability of **X** existing on the list after **n**. Note that $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}, \mathbf{x_5}, \ldots \mathbf{x_k} \ldots$ are all digits from **0** to **9** inclusive. The probability of *any* permutation of digits from **0** to **9** inclusive existing in a list of such permutations of digits is non-zero. Thus no matter how large $\mathbf{k}$ gets, the probability of **X** = $\mathbf{0.x_1x_2x_3x_4x_5} \ldots \mathbf{x_k} \ldots$ existing on the list *after* **n** must be a small but *non-zero* number.

But since the list of naturals is assumed to be infinite, the probability of the decimal number **X** = $\mathbf{0.x_1x_2x_3x_4x_5} \ldots \mathbf{x_k} \ldots$ existing in this infinitely large part of the list after **n** *must be 100%!*

That is because given an *infinite* number of chances of something happening, no matter how small the probability of it happening just *once,* as long as its probability of happening even once is not *absolutely* zero, it *must* happen if there are an infinite number of chances of it happening.

It might be argued that $\mathbf{x_k}$ is not the *last* digit in **X** = $\mathbf{0.x_1x_2x_3x_4x_5} \ldots \mathbf{x_k} \ldots$, and that there are an infinite number of digits *after* $\mathbf{x_k}$. This is true, but even so, the probability of the decimal number **X** = $\mathbf{0.x_1x_2x_3x_4x_5} \ldots \mathbf{x_k} \ldots$ existing on the list after **n** is *infinitesimally* small; but small as it is, it is *still not zero.*

And by the laws of probability, a non-zero probability of anything happening, given an infinite number of chances for it to happen, requires us to concede that it *must* happen.

If **X** = $\mathbf{0.x_1x_2x_3x_4x_5} \ldots \mathbf{x_k} \ldots$ *must* exist on the list after **n**, then given an *infinite* number of naturals, a natural *can* be found in this part of the list — namely *after* **n** — which *can* be put in one-to-one correspondence with it …

... Q.E.D. (And this time for real!)

4. The Clinching Argument

It is to be noted that if we adapt Cantor's proof to natural numbers, we can "prove" that natural numbers *themselves* cannot all be put in one-to-one correspondence with (other) natural numbers! Talk about a *reductio ad absurdum.*

Note that if we put all the natural numbers in the left column of a table, and then put other natural numbers in the right column, just as we have done for Table 1, we would get Table 2, as follows:

Table 2

Natural numbers	Other Natural Numbers
1	$D_1 = d_{11}d_{12}d_{13}d_{14} \cdots d_{1k} \cdots$
2	$D_2 = d_{21}d_{22}d_{23}d_{24} \cdots d_{2k} \cdots$
3	$D_3 = d_{31}d_{32}d_{33}d_{34} \cdots d_{3k} \cdots$
4	$D_4 = d_{41}d_{42}d_{43}d_{44} \cdots d_{4k} \cdots$
…	
n	$D_n = d_{n1}d_{n2}d_{n3}d_{n4} \cdots d_{nk} \cdots$
(etc)	

Here, **D** now represents any natural number, **d** again represents any digit between **0** and **9** inclusive, the first subscript of any particular **d** is the natural number to which that particular **d** corresponds, and the second subscript of that same **d** is the number of places that particular **d** lies to the right of the starting digit of the particular **D** to which that particular **d** belongs. (When we say "starting digit" we mean the first digit one normally writes down when one begins writing the natural number in question.)

Now each row of the table represents, of course, a natural number put in one-to-one correspondence with another natural number.

Now consider the natural number $X = x_1x_2x_3x_4x_5 \ldots x_k \ldots$, where x_1 is any digit other than d_{11}; x_2 is different from d_{22}; x_3 is not equal to d_{33}; x_4 is not d_{44}; and so on. Now, **X** is a natural number, so it should be in our list. *But where is it?*

The natural number **X** can't be the *first* in the list, since the first digit of **X** differs from the first digit of $\mathbf{D_1}$. Similarly, **X** can't be *second* in the list, because **X** and $\mathbf{D_2}$ have different second-place digits; and **X** can't be *third* in the list, because **X** and $\mathbf{D_3}$ have different third-place digits. In general, **X** can't be the $\mathbf{n}^{\text{th}}$ in the list — *i.e.,* it cannot be equal to $\mathbf{D_n}$ — since their $\mathbf{n}^{\text{th}}$ digits are not the same.

The natural number **X** is *nowhere* to be found in the list, no matter how large **n** gets. In other words, we have exhibited a natural number that *ought* to be present in a huge humongous giganto list — but it *isn't.* No matter how we try to list the natural numbers, and how long the list gets, at least *one* natural number will be left out.

Therefore, using Cantor's *own* argument, we have "proved" that putting the natural numbers in one-to-one correspondence with the natural numbers *themselves* is impossible!

But this is utterly absurd, and therefore something must be terribly wrong with Cantor's so-called "proof".

(Actually, what's *really* absurd is the way people keep on repeating Cantor's argument, and affirming it to be a solid cast-iron proof, virtually *ad nauseam,* in all the high school, college and university mathematics textbooks — and even in prestigious volumes such as those of the major Encyclopaedias. One wonders where their authors' and editors' heads are at!)

Acknowledgements

G. Walton's excellent Web page <http://www.btinternet.com/~sapere.aude/page2.html#ca> was the inspiration for the above argument.

I was also inspired by the page entitled *"Problems with Cantor's Diagonal Method and Infinity in General"* by Daniel Grubbs-Saleem. This page used to be at <http://users.javanet.com/~cloclo/infinity.html>,but it seems to have been taken down from the internet (as of July 2012).

Comments

Any comments? E-mail me: `ardeshir [at] mac.com`.