On the Gödel's formula

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Abstract

The proof of Gödel's first incompleteness theorem includes the construction of an arithmetic formula G that represents the metamathematical statement: the formula G is not provable. This article examines the formula G (of Gödel). We demonstrated that the Gödel's number of the formula G is not a finite number if (i) G is comprehended as a self-referential statement or (ii) there is an infinite set $\mathbb{S}$ of well formed formulae with one free variable such that the elements of $\mathbb{S}$ are theorems or antitheorems in T.

1 Introduction

The Gödel's formula is frequently comprehended as a self-referential statement like

This sentence is not provable.
Let G be the name of the above sentence.

In the section **2** we show that the Gödel's statement can not contain its own Gödel's numeral. Since in the indirect self-reference the formula refers to itself through of one or more formulae, the conclusion for direct self-reference is extended to indirect self-reference cases when Gödel's number of a formula is used as the name of the formula.

Section **3** examines the Gödel's formula without considering self-reference. Starting from the diagonal theorem we conclude that if there is an infinite set $\mathbb{S}$ of well formed formulae with one free variable such that the elements of $\mathbb{S}$ are theorems or antitheorems in T then the Gödel's sentence does not exist.

2 Gödel's formula and self-reference

A category of names of formulae, denominated *structural-descriptive names* by Tarski [1], is applied to names that describe the words that compose the denoted expression.

Gödel's numbering form attributes a distinctive numeral to each symbol of the alphabet of a formal language. It possesses an effective method to map each symbol, sequence of symbols (which can be a well formed formula) or sequence of well formed formulae (which can be proof of a theorem) in a numeral (denominated Gödel's number), and it possesses an effective method to map each Gödel's number in the symbol or sequences of symbols corresponding to the Gödel's number.

It is evident that the Gödel's number of a formula is a structural-descriptive name of the formula. The formula named G has a second name that is its Gödel's number. We will now build the formula G.

Let y be the Gödel's number of a well formed formula with a single free variable, z

$$y \qquad\qquad \Delta z \Omega \tag{1}$$

where Δ and Ω are sequences of symbols.

Let us define the function $sub(y, z, j)$ as being the formula obtained with the substitution, in the formula of Gödel's number y, of the only free variable, z, by the number j. The sequence $sub(y, z, j)$ is

$$\Delta \mathsf{sss}\ldots\mathsf{s}0\Omega$$

where s means “the successor of” and there are $j+1$ characters in the sequence $\mathsf{sss}\ldots\mathsf{s}0$. Let us denote by $g(A)$ the Gödel's number of the formula A and by $\ulcorner A \urcorner$ the sequence $\mathsf{sss}\ldots\mathsf{s}0$ of $g(A)+1$ symbols. Let us consider the formula

$$\neg(\exists r : \exists s : (P(r,s) \wedge (s = \ulcorner sub(y,z,y) \urcorner)) \tag{2}$$

$P(r,s)$ is true if the sequence of symbols with Gödel's number r proves the formula with Gödel's number s. In English (2) means: *there is not a proof for the formula whose Gödel's numeral is* $\ulcorner sub(y,z,y) \urcorner$.

For us to derive the Gödel's number of (2), we substituted $\ulcorner sub(y,z,y) \urcorner$ by the sequence $\mathsf{sss}\ldots\mathsf{s}0$ of $g(sub(y,z,y))+1$ symbols

$$\neg(\exists r : \exists s : (P(r,s) \wedge (s = \mathsf{sss}\ldots\mathsf{s}0)) \tag{3}$$

Let n be the Gödel's number of (2). Assuming the variable y of function $sub(y,z,y)$ the value n, the formula (2) changes to

$$\neg(\exists r : \exists s : (P(r,s) \wedge (s = \ulcorner sub(n,z,n) \urcorner)) \tag{4}$$

The formula (4) is also $sub(n,y,n)$ and its Gödel's number is

$$g(sub(n,y,n)) \tag{5}$$

If we know y, then we know the corresponding formula and the position of the free variable z in it. The character z in (2) is determined and substituted by y. In this sense the free variable in (2) is y.

In the presentation with the name of the formula and the formula, we have

$$\underbrace{g(sub(n,y,n))}_{\text{name of the formula}} \qquad \underbrace{\neg(\exists r : \exists s : (P(r,s) \wedge (s = \ulcorner sub(n,z,n) \urcorner)))}_{\text{formula}} \tag{6}$$

The proof of Gödel's first incompleteness theorem includes the construction of an arithmetic formula G that would represent the metamathematical statement: formula G is not provable. This formula G is in Hofstadter [2] or Nagel and Newman [3], for instance,

$$\underbrace{g(sub(n,z,n))}_{\text{name of the formula}} \qquad \underbrace{\neg(\exists r : \exists s : (P(r,s) \wedge (s = \ulcorner sub(n,z,n) \urcorner)))}_{\text{formula}} \tag{7}$$

To come to (7) the symbol z needs to be understood as the free variable in (2) and Gödel's number of (4) defined by

$$g(sub(n,z,n))$$

Let us notice that

$$\underbrace{g(sub(n,z,n))}_{\text{name of the formula}} \qquad \underbrace{\neg(\exists r : \exists s : (P(r,s) \wedge (s = \ulcorner sub(y,z,n) \urcorner)))}_{\text{formula}} \tag{8}$$

is obtained if z is understood as the free variable in (2) and $\ulcorner sub(y,z,y) \urcorner$ in (2) is substituted by $\ulcorner sub(y,z,n) \urcorner$. The formula presented in (7) needs to be examined:

Theorem 2.1 *The Gödel's number of the formula G is not a finite number.*

Proof:

Let be the equivalent sets

$$\{(r_1, s_1), (r_2, s_2), (r_3, s_3), \ldots, (r_k, s_k)\} \tag{9}$$

and

$$\{(1, 2), (3, 4), (4, 6), \ldots, (2k - 1, 2k)\} \tag{10}$$

where r_i is Gödel's number of the proof of the formula of Gödel's number s_i. Let's consider that the number of provable formulae, denoted by k, is finite. Let be now the set

$$\{0, (1, 2), (3, 4), (4, 6), \ldots, (2k - 1, 2k)\} \tag{11}$$

let us make correspond one-to-one to element 0 of the set given in (11) the Gödel's number $g(G)$. It is evident that, by hypothesis, G is not a provable formula (there are k provable formulas and G is not one of them). Therefore, G can be built as

$$\underbrace{g(G)}_{\text{name of the formula}} \qquad \underbrace{\neg(\ulcorner G \urcorner = \ulcorner s_1 \urcorner) \wedge \neg(\ulcorner G \urcorner = \ulcorner s_2 \urcorner) \wedge \ldots \wedge \neg(\ulcorner G \urcorner = \ulcorner s_k \urcorner)}_{\text{formula}} \tag{12}$$

where $\ulcorner s_k \urcorner$ is the Gödel's numeral correspondent to s_k. If $g(G)$ is finite, then just the contribution of the first Gödel's numeral to count from the left in the formula in (12), for Gödel's number of the formula, is greater than the name of the formula in (12). This is a contradiction. Therefore the Gödel's number of the formula G is not a finite number. □

3 Gödel's formula and the diagonal theorem

The diagonal theorem, which is the core of Gödel's proof, says [4]

Theorem 3.1 *For any well-formed formula (wf) D(x) with x as its only free variable, there exists a closed formula α such that*

$$\vdash_T \ \alpha \longleftrightarrow D(\ulcorner \alpha \urcorner) \tag{13}$$

The formula $D(x)$ used in Gödel's proof is

$$D(s) = (\forall r)\neg P(r, s) \tag{14}$$

Substituting (14) in (13) we obtain

$$\vdash_T \ G(x) \longleftrightarrow (\forall r)\neg P(r, \ulcorner G(x) \urcorner) \tag{15}$$

Let

$$\mathbb{S} = \{s_1, s_2, s_3, \ldots, s_k, \ldots\} \tag{16}$$

be the set of all well formed formulas with one free variable that are theorems or antitheorems in T. The set $\mathbb{S}$ is not finite. We can construct the following sequence of symbols, denoted by γ,

$$\neg(\ulcorner Z \urcorner = \ulcorner s_1 \urcorner) \wedge \neg(\ulcorner Z \urcorner = \ulcorner s_2 \urcorner) \wedge \ldots \wedge \neg(\ulcorner Z \urcorner = \ulcorner s_k \urcorner) \wedge \ldots \tag{17}$$

Let be the Gödel's numerals of γ substituted by the correspondent sequences of $\mathsf{sss}\ldots\mathsf{s}0$. The sequence γ says with symbols of the list of symbols of T that Z(x) is not provable. There are infinite symbols in γ and there is no way to express what γ says with a number finite of symbols. Therefore the formula

$$(\forall r)\neg P(r, \ulcorner G(x) \urcorner) \tag{18}$$

has not a Gödel's number (the number obtained is not finite) and it is not a well formed formula. With this we proved the following theorem:

Theorem 3.2 *If there is an infinite set $\mathbb{S}$ of well formed formulae with one free variable such that the elements of $\mathbb{S}$ are theorems or antitheorems in T, then the Gödel's sentence does not exist.*

References

[1] A. Tarsky, *The Semantic Conception of Truth and the Foundations of Semantics*, Philosophy and Phenomenological Research **4** (1944).

[2] D. R. Hofstadter, *Gödel, Escher, Bach: an Eternal Golden Braid*, Vintage Books, (1980).

[3] E. Nagel and J. R. Newman, *Prova de Gödel*, Editora Perspectiva, (1973).

[4] Noson S. Yanofsky, *A Universal Approach to Self-referential Paradoxes, Incompleteness and Fixed Points*, arXiv:math.LO/0305282 v1 19 May 2003.