ESSAY:
AN ATTEMPT TO SOLVE THE "LIAR PARADOX"

by

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CONTENTS

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INTRODUCTION

The "Liar Paradox" was first discerned, as far as is known, in ancient times with Epimenides the Cretan, who stated: "All Cretans are liars". Since Epimenides was himself a Cretan, it implied that Epimenides was himself a liar; and if he was a liar, what he had just said, namely: "All Cretans are liars", could not be true. But if that sentence — now widely known as "The Liar Sentence" or simply "The Liar" — were not true, then Epimenides was *not* lying when he said "All Cretans are liars", and thus must have been telling the truth. As a result, if Epimenides was telling the truth it implied that he was lying, while if he was lying it implied that he was telling the truth. The result is a paradox.

"The Liar" has many variations. The one Bertrand Russell was stumped with when trying to formulate his *Principia Mathematica* in collaboration with Alfred North Whitehead, was the version wherein on one side of a sheet of paper is written the sentence: "The sentence on the other side of this paper is true", while on the other side of the paper is written the sentence "The sentence on the other side of this paper is false". If the first sentence is taken as true, then the second must be taken as true too, in which case the first turns out to be false; whereas if the first sentence is taken as false, then the second one is to be taken as false too, in which case the first turns out to be true.

RUSSELL'S "THEORY OF TYPES"

To solve this paradox and others like it, Russell formulated his "Theory of Types". In this theory, Russell divides all sentences in to a hierarchy of "types", and prohibits a sentence of one type from being used to validate or invalidate a sentence of another type. In Russell's view, all objects for which a given condition (or predicate) holds must be at the same level or of the same "type".

The argument to justify this division into "types" is, according to Russell, that in each case a totality is assumed such that if it were legitimate it would at once be enlarged by new members defined in terms of itself. Hence he adopts the rule:

Whatever involves all of a collection must not be one of the collection.

Unfortunately for logic, the above rule is more fully explicated as:

If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total.

This is the "vicious circle" principle, and its consequences for the development of logic are considerable. As R.B. Jones writes in this regard, at the Web site denoted by the URL <http://www.cybercom.net/users/rbjones/rbjpub/philos/bibliog/russ08.htm#SI>: "Though Russell has undoubtedly put his finger on the spot, the spot is a great deal smaller than the finger, and his rule obliterates much of importance."

PARTIAL "LIAR" PARADOXES

Other versions of "The Liar" are also widely known. One version, often used and even accepted in philosophical writings tending towards mysticism such as Zen Buddhism, is the sentence:

The ultimate truth cannot be put into words.

Note that if this is true, it being itself in words, it *cannot* be true. However, this sentence does not entail a *totally* paradoxical conclusion, because if it is itself taken as *false* rather than true, then no paradox is generated. Thus it is only a partial version of "The Liar".

Until now "The Liar" has been hard to crack. As was expressed by an e-correspondent of mine, Dr Joseph S Fulda, in a recent e-mail, "There is no agreed-upon answer to the Liar. Some say it is neither true nor false, some say it is both, some say it is a category mistake to speak of its truth, some say it is meaningless, some say it is logically false, etc."

PRESENT ATTEMPT AT SOLVING "THE LIAR PARADOX"

The present essay attempts to give a solution to "The Liar Paradox" such that not only is the original paradox removed, but also partial paradoxes such as the one generated by saying "The ultimate truth cannot be put into words". It also bases itself on common sense, so that the solutions proposed fit in with "the real world", and are not restricted to strictly logical formats.

Although the present approach makes use of many of the same arguments as fuzzy logic, it is to be noted that it also goes beyond fuzzy logic, in that it attempts to unify fuzzy logic with standard (i.e., non-fuzzy) logic.

In describing the present approach, it is perhaps best to start with propositions the truth of which is irrefutable, and work onward from there. (The word "proposition" is used in preference to the word "sentence" because the same proposition can be expressed by several different sentences, and after all it's the thought that counts — in logic as much as in gift-giving!) ☺

IRREFUTABLE PROPOSITIONS, TRIVIAL AND NON-TRIVIAL

That there are *any* propositions whose truth is irrefutable at all has been widely disputed, especially in post-modernist philosophy. However, to say there are no such truths involves the same partial paradox as saying "The ultimate truth cannot be put into words", and as a result, to say that there aren't any irrefutable propositions can't be true, for if it were true it would deny itself. Moreover, examples can easily be given of irrefutable propositions.

Take the case in which a proposition — *any* proposition — has been enunciated. In that case, the following further proposition (call it "proposition [A]"):

[A]   A proposition has been enunciated

… *must* be true: right? It cannot *but* be true. Its truth is irrefutable.

Similarly, take the case of a doubt arising as to the truth of a proposition: again, *any* proposition. In that case, the following further proposition (call it "proposition [B]"):

[B]   A doubt has arisen as to the truth of a proposition

… *must*, again, be true, and cannot be false.

Or take propositions that involve definitions. Suppose an oxygen atom is defined as "an atom having eight protons in its nucleus"; in that case, the proposition (call it "proposition [C]"):

[C]   The oxygen atom has eight protons in its nucleus

… *must* be true, by *definition* — and thus must be irrefutable.

The objection may be put forward that these are trivial examples of irrefutable propositions, and that is readily admitted; but the point to be noted here is that they do exist, even if they are admittedly trivial.

However, as a result of them existing, the proposition (call it "proposition [D]"):

[D] *Irrefutable propositions do exist*

… *must* also be true!

Now note that [D] is *not* a trivial proposition by any means. Thereby it is also proven that the following proposition (call it "proposition [E]"):

[E] *At least one non-trivial irrefutable proposition exists*

… must also be irrefutably true.

Since [E] is also non-trivial, from here on it is easy to generate the proposition (call it "proposition [F]"):

[F] *Non-trivial irrefutable propositions do exist*

… which must also be irrefutably true.

PROPOSITIONS WHICH ARE IRREFUTABLY FALSE

It is to be noted that if there are propositions that are irrefutably true, there must also be others that are irrefutably false. For example, take the following proposition (call it "proposition [C']"):

[C'] *The oxygen atom does* ***not*** *have eight protons in its nucleus*

… must be irrefutably false, since by definition, the oxygen atom *does* have eight protons in its nucleus.

FALSE PROPOSITIONS WHICH ARE APPROXIMATELY TRUE

Now in addition to the propositions the truth of which is irrefutable, and of which examples have been given above, there are also other propositions which, though not strictly speaking true, are also close enough to the truth as to make little difference.

For example, if a book were purchased for $19.99, it would strictly speaking be false to say twenty dollars were paid for it, but even the IRS might forgive you if you indicated so small a discrepancy on your tax return.

Or similarly, to say that the value of π is 3.141592653589793 is strictly speaking false, but few engineers will quibble about it. (And note that it doesn't matter to how many decimal places π is worked out; as long as it is expressed as a decimal number, that number can only be *approximately* equal to π, never *exactly* equal to it.)

In the sciences many such examples are found. For example the proposition (call it "proposition [G]"):

[G] Water boils at sea level at 100 degrees Celsius

… is strictly speaking false, for the boiling point of water depends, among other things, on atmospheric pressure, and even at sea level atmospheric pressure varies from day to day and at times from hour to hour; but [G] is all the same close enough to the truth to be used as a "scientific law".

POPPERIAN "FALSIFIABILITY"

This is one reason why Sir Karl Popper has enunciated the notion that science advances, not by enunciating laws that are *true*, but by enunciating laws that are not yet "falsified". For example proposition [G] is not strictly speaking true, and thus has been "falsified". In other words, the following proposition (call it "proposition [G']"):

[G'] Water does not necessarily boil at sea level at 100 degrees Celsius

… *is* strictly speaking true, since it is the same as saying that proposition [G] is false.

According to this view, positive empirical statements enunciated in the form of "scientific laws" cannot ever be true, but can only be as-yet-unfalsified. This is because of the impossibility of universal testing. Examples are: "All crows are black", or "All tigers have stripes". If even one non-black crow is found, that would falsify the first of these, and similarly, if even one tiger without stripes is found, that would falsify the second. And once falsified, their falsehood is definite and irrefutable.

However, even Popperian falsifiability cannot get round the truth of positive statements such as proposition [C], which are true by *definition*.

Now in criticism of the Popperian view, it is to be noted that such irrefutably true positive propositions were arrived at following reasoning containing propositions which, in the Popperian view, were strictly speaking false! For example, the definition of the oxygen atom as one which has eight protons in its nucleus was made only after arriving at the discovery of oxygen, of the atom, of the atomic nucleus and of the proton, all of which were at one time undiscovered. Without these previous discoveries, it would not even have been possible to enunciate the *definition* that an oxygen atom is one whose nucleus has eight protons! The existence of that very definition is based on discoveries previously made.

And yet, in the Popperian view, all of these previous discoveries were at one time empirical, and thus based on propositions which, in the strictly Popperian view, must have been false: if not demonstrably, then at least potentially (in the sense of being "falsifiable".)

So a conundrum is generated, namely: how can reasoning based on false or potentially false propositions end up yielding a proposition which is *irrefutably* true?

APPROXIMATIONS

This conundrum is resolved if we realise that even though the propositions which led to the final enunciation of the irrefutably true proposition "All oxygen atoms have nuclei containing eight protons each" were *strictly* speaking either false or falsifiable, they could only have been false, even when actually falsified, by a very small amount: much like saying "This book costs twenty bucks" when the price of the book is in reality $19.99. The propositions, in other words, which led to the irrefutable truth of the proposition "All oxygen atoms have nuclei containing eight protons each" were *approximately* true.

If *approximately* true propositions are allowed, then Popperian "falsifiability" can be amended to say that most of the laws of science consist of propositions which are *approximately* true, even if not *strictly* true. This explains why science can not only yield significant results in the "real" world, but can even ultimately lead to the enunciation of propositions which are irrefutably true. This is the approach of fuzzy logic.

Note also that approximations can either be close or not so close. For example, in another bookstore the same book which was purchased for $19.99 might be marked for sale at $19.95. This is not quite as close an approximation to "twenty dollars" as the price of $19.99; however, even if the book were purchased for $19.95, the purchaser might be justified in saying to his friends "I got it for a measly twenty bucks".

Thus it might be justifiable to say that science advances by enunciating "laws" made up of propositions which are demonstrably closer to the truth than those which they supplant.

APPROXIMATELY FALSE PROPOSITIONS

If there are propositions which are approximately true, then there must also be propositions which are approximately false (i.e., not *entirely* false.) For example, if the above-mentioned book were purchased for either $19.99 or $19.95, it would be *entirely* false to say that it was not purchased at all; but it might be *approximately* false to say that it was purchased for, say, $15.00; and it would be even more false to say that it was purchased for $10.00, and it would be yet more false to say that it was purchased for just one dollar.

WIDE RANGE OF TRUTH VALUES

The above indicates that leaving aside propositions that are either irrefutably true nor irrefutably false, there is a wide range of truth values that can be applied to propositions. Some are very close to the truth — perhaps so close to the truth that it makes next to no practical difference (like the value of π worked out to ten billion decimal places, which apparently has been done!); on the other hand, some are just barely squeaking by even as approximate truths (like saying "Politicians, by and large, are not inveterate liars"); and yet others are rather clearly false, though not entirely so, like the little kid who says "But Mom, I did brush my teeth" (yeah, right, and when was that: a week ago?).

This, as mentioned earlier, is the principle used in fuzzy logic. And this is why fuzzy logic works so well in practical applications; for most of the statements made in so-called "real life" are of the kind whose truth value is approximate. Confidence tricksters know this very well, when they tell tales that are by and large true enough, and are only untrue in some small detail (which however turns out in the end to be the detail that matters the most!)

STATEMENTS THAT ARE NEITHER TRUE NOR FALSE

Of course there are also statements that are neither true nor untrue: namely those which are meaningless or nonsensical. Of course some statements are rather clearly meaningless: for example, the statement "pwioz eruiytr siuyu btyer xueiw qoiunlnkh" is not only meaningless, but even unpronounceable: it is, in fact, utter gibberish. (People who mumble their sentences make just such noises: politicians, especially when asked embarrassing questions by journalists, are rather good at this sort of non-communication!)

It is to be noted however that a statement which just *sounds* like gibberish isn't *necessarily* so. For example the sentence "Questo libro è stato scritto con l'intenzione di fornire informazioni al pubblico" in *not* gibberish to someone who understands Italian, but it *is* meaningless or nonsensical to someone who doesn't.

But some statements, even when clearly enunciated in a language that is well understood, are meaningless, or at least nonsensical: such as the statements "A stitch in time is worth two hands in the bush" or "A rolling stone is only skin deep". Grammatically these statements are correct, but they make no good sense even to those who understand English perfectly well. (It is to be noted that here we use the word "statements" instead of "propositions" because there is really no cogent or meaningful *thought* behind the words.)

Now these are rather clear examples of meaningless or nonsensical statements, but there are others whose meaninglessness is very subtle, and rather hard to pin-point. This is especially so for tricky questions. For example, the questions " Where does a fire go when it goes out?" (an ancient question found in Hindu sacred texts) or "Where does a fist go when the hand is opened?" (a well known Zen *koan*) can only be answered with another question, such as "What exactly is meant here by the word 'go'?", or else they can be answered by saying "Such questions are meaningless". They do not admit meaningful answers because they are based on a conceptual confusion of two distinct logical concepts. The questions thrive on the mistaken syntactical similarity with another question like "Where do I go when I fall ill?", to which an appropriate answer might be "I go to the hospital". The last question is perfectly meaningful while the first two are incomprehensible; the last admits of a perfectly meaningful answer while the first two do not. They are questions which are suggested by the grammar of the language but which give or imply a false or distorted picture of the nature of reality. Here too, there can be no meaningful *thought* behind the words.

It is to be noted that it is not always possible to pin-point exactly what makes a statement meaningless. This is likely because the meaning of the word "meaning" is itself hard to pin-point exactly.

DEGREES OF MEANINGLESSNESS

Also, as can be seen above, there are degrees of meaninglessness. The statement "pwioz eruiytr siuyu btyer xueiw qoiunlnkh" is *utterly* meaningless: indeed it is almost not even worth calling it a statement. On the other hand, the sentence "Questo libro è stato scritto con l'intenzione di fornire informazioni al pubblico" is *not* meaningless, even though a person who does not understand Italian might not get its meaning; whereas the statement "A stitch in time is worth two hands in the bush" *is* meaningless even to speakers of English, and even though it is perfectly good grammatically.

Note also that it is quite conceivable that a code or a language can be devised in which the jumble of letters "pwioz eruiytr siuyu btyer xueiw qoiunlnkh" *is* really a meaningful statement. It is just meaningless as it stands.

Thus extending the view that some propositions are irrefutably true while others are approximately true to statements that are meaningless, one might go on to say that some statements are utterly meaningless while others are only approximately so. And as with truths, one might say that some approximations are closer to utter meaninglessness (or, for that matter, to utter meaningfulness!) than others.

"THE LIAR" CONSIDERED IN LIGHT OF THE ABOVE

The above considerations now may be applied to "The Liar." If, instead of arguing that a proposition or statement can only be true, false or meaningless, one extends the argument to say that a proposition or statement should be considered either irrefutably true or irrefutably false, approximately true or approximately false, or utterly meaningless or approximately meaningless, then before one judges the truth or otherwise of a statement or a proposition, one ought to ask: in which of these categories is the statement or proposition? For without such categorising prior to arguing further from it, the conclusion drawn may be far off the mark.

Suppose "The Liar" specifically as enunciated by Epimenides the Cretan is first considered. The statement he made was:

All Cretans are Liars.

If this statement is taken as *approximately* true — that is to say in the same category as "This book cost twenty bucks" when it actually costs only $19.99 or even $19.95 — then it does allow for one or two exceptions here and there. Epimenides might say such a thing with as much honesty as he might have said "I bought the book for twenty dollars" when he actually paid a nickel less. Thus in the "approximately true" category, Epimenides' "Liar Sentence" would imply that all Cretans might well be liars, but once in a while even one of them might be telling the truth. The paradox is thus removed.

ST PAUL'S UNDERSTANDING OF "THE LIAR"

Indeed this seems to be the way St Paul understands Epimenides. As he writes in the *Book of Titus*:

> Even one of their own Prophets has said, "Cretans are always liars, evil brutes, lazy gluttons." This testimony is true. Therefore, rebuke them sharply, so that they will be sound in the faith … . *(Titus 1:12 ff).*

Thus when St Paul writes above "This testimony is true", he obviously means "It is *approximately* true", not "It is *always* true in each and every case, including the one mentioned by the Cretan 'Prophet'."

(It is to be noted that there is no translation error here, since the original New Testament Greek here for "true" reads *alethes*, which indeed means "true" in New Testament Greek.)

Similarly, consider the version of "The Liar" which reads:

The ultimate truth cannot be put into words.

In this case too, if the statement above is taken to be *approximately* true, it does not generate a paradox. Thus it does not necessarily have to be taken as utterly false in order for the paradox to be removed. (This explanation should satisfy the mystics and Zen Buddhists!)

BERTRAND RUSSELL'S VERSION OF "THE LIAR"

Now consider Bertrand Russell's version, namely the piece of paper on one side of which is written the sentence: "The sentence on the other side of this paper is true", while on the other side of the paper is written the sentence "The sentence on the other side of this paper is false". In this case, the paradox is not *immediately* removed by taking the truth of the statements to be approximate.

However, it is to be noted that Russell's version when taken *precisely* as Russell took it ends up as a vicious circle. Now if instead, each of the statements on the paper is instead taken as only *approximately* true, then over a number of "cycles", so to speak, of this "circle", it renders each of the statements fairly innocuous, and the paradox ultimately gets removed.

In other words, in Russell's original version, the two statements end up in a "vicious circle"; in the amended version, the "circle" becomes a kind of spiral, which in the end provides an "out" for the elimination of the paradox.

To see how this happens, let us add the word "approximate" to each of the statements. Then the statements become, respectively:

"The sentence on the other side of this paper is approximately true",

… and

"The sentence on the other side of this paper is approximately false".

Remember that the word "approximate" means "close to but not exactly". As was noted earlier, there are degrees of approximation, and not all approximations are equally close to the absolute truth. Some approximately true propositions are closer to the absolutely true than are others.

Now beginning with the first of the two "Liar" statements quoted above, it is seen that the second statement is only approximately true, which, coupled with the word "approximately" in the second statement, makes the first statement only *approximately* approximately false. In other words, the word "approximate" in the first sentence decreases the closeness to the absolutely true of the first sentence, and since the word "approximately" also exists in the second sentence, this decreases still further the closeness to the absolutely true of the second, and so on.

To express it in the form of a loose analogy, it is as if the first statement were to say "The book costs $19.95" when it actually costs $19.99; then the second statement comes back with "No, it's worth only a dime". The first statement then retorts, as it were, "Okay, I'll let it go for $19.50", and the second statement comes back with a counter offer: "Well, let's bump it up to fifty cents and it's a deal." In the end the two agree on ten dollars or thereabouts, which however is a far cry from twenty!

As can be seen, the "vicious circle" has become a not quite so vicious "spiral". In other words, the paradox, although not removed right away, does end up being removed in the long run.

OBJECTIONS TO FUZZY LOGIC

As can be seen, most if not all of the principles used in the present approach use fuzzy logic. Now although fuzzy logic has been very successful in applications, it does not seem to have had much of a following among philosophers. In this regard, in fact, there have perhaps been more detractors than supporters (perhaps because of the association with the word "fuzzy" — like, "Hey, if you're in favour of fuzzy logic, your own logic must be f***ing fuzzy, what?"). As James F. Brulé indicates in *Fuzzy Systems — A Tutorial* (see <http://www.ortech-engr.com/fuzzy/tutor.txt>), while there have been generic complaints about the "fuzziness" of the process of assigning values to linguistic terms, perhaps the most cogent criticisms come from Haack (see S. Haack, "Do we need fuzzy logic?" Int. Jrnl. of Man-Mach. Stud., Vol. 11, 1979, pp.437-445.) Haack, who is a formal logician, argues that there are only two areas in which fuzzy logic could possibly be demonstrated to be "needed," and then argues that in each case it can be shown that fuzzy logic is unnecessary.

The first area Haack pin-points is that of the nature of truth and falsity: if it could be shown, she admits, that these are fuzzy values and not discrete ones, then a need for fuzzy logic would have been demonstrated.

However, Haack argues that "true" and "false" are discrete terms. For example, "The sky is blue" is, in her view, either true or false; any fuzziness to the statement arises, according to her, from an imprecise definition of terms, not out of the nature of truth.

The other area she identifies is that of fuzzy systems' utility: if it could be demonstrated that generalising "classical" logic to encompass fuzzy logic would aid in calculations of a given sort, then again fuzzy logic would be necessary.

However, she also argues that data manipulation is in no way made easier through the introduction of fuzzy calculus; if anything, she argues, the calculations become more complex. Therefore, she argues, fuzzy logic is unnecessary.

PHILOSOPHICAL ARGUMENTS AGAINST HAACK

The first of Haack's objections is fairly easily dealt with from a philosophical point of view. Although "true" and "false" are indeed discrete terms in some cases, then are certainly not so in all. Nor is it possible to define terms precisely in all cases.

To give an example of an approximate truth which cannot be further broken down into discrete truth by any more precise definition of the terms contained in it, take the statement:

It is difficult for me to run a mile or more.

The difficulty here lies in the word "difficult". Just when does easy running turn into difficult running?

Or take the sentence:

This activity has become painful, though it was pleasant a while ago.

At just what millisecond in time did the activity turn from pleasant to painful?

Innumerable sentences such as these can be found. Some of them are extraordinarily important ones — or are at least considered so by billions of the world's people. A prominent example is:

Thou shalt love the LORD thy God with all thy heart and all thy soul and all thy might.

The difficulty is in knowing with precision when that stage has been reached!

(Admittedly this last example is an imperative, and thus strictly speaking cannot bear a truth value; but a truth-value-bearing affirmative statement can easily be generated from it while preserving the original intent: such as the statement "It is true that one should love the LORD one's God with all one's heart and all one's soul and all one's might".)

IMPOSSIBILITY OF PRECISELY DEFINING ALL TERMS

Considering the above examples, it is clearly impossible to define with precision when difficulty starts, a pleasurable activity becomes painful, or love has reached its limit. (If it could be done, don't you think some smart logician would have by now put Moses's words in a more accurate format?)

And yet there can be no question that the above statements are capable of being true or false. They are certainly not meaningless.

Even the colour "blue", as in "The sky is blue" — which Haack seems to think can be defined with precision — cannot really be defined with precision without altering the everyday meaning of the word "blue". If "blue" is defined as light of wavelength "x", then would one have to call light of wavelength "x-minus-a", when "a" is a very, very tiny amount, to be "not-blue"? That sounds ludicrous.

Even if "blue" were defined as light of a *range* of wavelengths from "x-to-y", a very tiny amount outside this range would have to be classified as "not-blue", which would not be in keeping with the way the word "blue" is used normally.

In other words, under a precise enough definition of "blue", a sky which might be called "blue" by any sensible human being might, under a strict enough definition, turn out to be "not-blue", which is ludicrous. Or else, if the range "x-to-y" were large enough, a sky which might under that definition be called "blue" would not be so called by any right-thinking human being.

Similarly, even if a painful experience were to be precisely defined (say in neuroscience terms), it would end up by some experiences which are *defined* as painful being *experienced* as not painful, and vice versa. That again is ludicrous: no one would want another person — or a machine — telling them what is painful and what is not!

And as for the second of Haack's objections, although it is true that the way fuzzy logic is at present expressed in calculations is more complex than the way "classical" forms of logic are expressed, that by itself doesn't render fuzzy logic unnecessary. If fuzzy logic can do what "classical" forms of logic can't — like cracking "The Liar" wide open — then the extra work is surely justified: just as the extra work required for Quantum Mechanics as compared with Newtonian Mechanics is justified.

Besides, fuzzy logic seems to be the way the mind works naturally. If so, maybe it will one day be possible to express the calculations of fuzzy logic in ways that are more natural than those used to express "classical" forms of logic.

ADVANTAGES TO THE PRESENT APPROACH

Taking the present approach as a whole — that is, as a combination of fuzzy logic with the additions and modifications noted above — it is seen that it has the benefit of fitting in with the "real world", and not just with pure logic. Note that in the earlier-mentioned analogy about the price of the book, had both the buyer and the seller stuck to their guns, not budging one red cent, there would never have been a sale, even for ten bucks. Ten bucks might not satisfy the seller as much as twenty, but surely no sale at all would have been even worse for him! And the buyer would have gone away empty-handed, which would be quite unsatisfactory for him too. With a compromise, even though neither of them are *totally* satisfied, at least they are also not *totally* dissatisfied.

And yet the present approach also fits in with logic, even symbolic logic. Fuzzy logic can, of course, be expressed in symbols. If in addition extra symbols were created for notions indicated in the present essay — such as "absolutely meaningless", "approximately meaningless", etc., etc. — it should be possible to enunciate the "truth value" of any version of "The Liar", in such a way as to remove many paradoxes.

The present approach may be likened, in fact, with carrying out surgery with an almost infinitely variable laser scalpel — while the traditional approach might be likened to carrying out surgery with a kitchen knife, or at best with an old-fashioned straight razor.

However, it is not claimed that the present approach can resolve *all* paradoxes: for example, I don't think it can resolve the paradoxes generated by assuming that time-travel is possible. However, other approaches might be able to resolve time-travel paradoxes: such as the argument — common to many Eastern philosophies — that the notion of time is itself based on things whose validity or existence is to some extent doubtful, namely mental memory and material records; and as a result, time might be considered to be a sort of illusion.

OTHER PARADOXES RESOLVED

Note however that there *are* other paradoxes that are resolved by taking this approach. One such is the paradox generated by saying words — as is indeed often said — to the effect that "it is impossible to secure universal agreement as to the meanings of words (or statements)". This is said to bolster the notion that there can be no universal truths. However, saying so generates a paradox, since if there can be no universal truths, then the statement:

It is impossible to secure universal agreement as to the meanings of words

… itself cannot be universally true; but in that case, it *may* be possible to secure universal agreement as to the meanings of words.

However, if the above sentence were amended to say:

*It is **almost** impossible to secure universal agreement as to the meanings of words*

… then there is no paradox.

THE SELF-DENIAL OF POPPERIAN FALSIFIABILITY

Perhaps more importantly for science, the above approach also resolves the self-denial generated by Popperian falsifiability. According to Popper, even though it is impossible to enunciate absolute empirical truths that are *positive,* it *is* possible to enunciate absolute empirical truths that are *negative.* For instance, according to Popper, if even one tiger without stripes can be found, it *is* possible to say absolutely truthfully "There *are* tigers that do not have stripes."

But as a result, Popperian falsifiability denies itself; for if it is itself absolutely true, by its own admission it should be impossible to absolutely and truthfully say:

*It **is** possible to enunciate absolute empirical truths that are negative.*

(Note that although the term "negative" does appear in this statement, it is an adjective, and not an adverb like "not" or "without". Thus the above statement is itself a *positive* one, not a negative one.)

But if on the other hand the term "approximate" — or "almost", which amounts to the same thing — is inserted into Popperian statements, then there is no self-denial. For example, it might be said:

*It is **almost** impossible to enunciate absolute empirical truths that are positive.*

In that case, this modified version of Popperian falsifiability does not deny itself.

RUSSELL'S PARADOX

Although paradoxes such as "The Liar" and the paradox of Popperian "falsifiability" are fairly easy to crack open using the present approach, it is admitted that the present approach finds it much tougher to crack the paradox known as "Russell's Paradox" (which is not to be confused with the way Russell enunciated "The Liar"). This paradox is generated when it is asked: "Is the set of all sets which are not members of themselves, a member of itself or not?" If it is, then by definition it cannot be a member of itself, while if it is not, then by the principle of exclusion, it must be a member of itself.

Now the concept of membership does not seem (at least at first blush) to allow for "approximate" membership. For instance, one can't "approximately" be a citizen of the United States! Either one is or one isn't. (On second thoughts, maybe not — what about "honorary citizens", like Sir Winston Churchill? Indeed fuzzy logic does allow for "degrees of membership": see Web page at <http://www-dse.doc.ic.ac.uk/~nd/suprise_96/journal/vol2/jp6/article2.html>).

Obviously there are degrees, for instance, of "in" and "out": for example, one can be partially in a room and partially out of it (as when just stepping into it.) However, the concept of degrees of membership does not seem to be applicable to *every* sort of membership: for example, either something is moving or it isn't; or to give a subjective example, either there is awareness or there isn't.

Thus the set of "All things that are in the room" does allow for degrees of membership, while the set of "All things that do not move" does not allow for it.

If "degrees of membership" are allowable in the case of "The set of all sets", then Russell's Paradox is also easily solved by using that concept. However, even if it be taken as strictly true that "either something is a member of a set or it isn't" — in which case the present approach cannot tackle Russell's Paradox *directly* — even then the present approach *can* tackle Russell's Paradox indirectly. To do this, it takes advantage of the fact that statements can be *approximately meaningless*.

Assuming that membership in a set cannot be approximate, if we take the question "Is the set of all sets which are not members of themselves, a member of itself or not?" to be *completely* meaningful, then we do end up with a paradox. However, if we take the question to be only *approximately* meaningful — that is to say if we take it that although it is grammatically correct it doesn't make a whole lot of sense, much like the question "Where does the fist go when the hand is opened?" — then it might be possible to answer it absolutely truthfully it by saying "There is no absolutely true answer to this question, with the sole exception of the present answer."

The notion of approximate meaningfulness or meaninglessness might be explained as being analogous to the way π has been calculated to ten billion decimal places. Even though such a calculation would be accurate at the sub-atomic range of dimensions even for a circle as large as the entire known universe, it is still, in theory at least, only approximate.

And it is also not known by just *how much* it is off the true value, because if that were known, then the calculation would be true, not to just ten billion decimal places, but to at least ten billion and one! Nevertheless it is still only approximate.

In an analogous fashion, it might be said that the question:

Is the set of all sets which are not members of themselves, a member of itself or not?

… is indeed very close to being totally meaningful, but is not all the way there; and indeed it is impossible to even say just by how much it is off, nor in just what way it is not quite meaningful; but nevertheless, since it is not *totally* meaningful, it does not permit a totally meaningful answer either. In this way the paradox is removed.

ADVANTAGES OF THE PRESENT APPROACH

It is to be noted, also, that even if it be asserted that the above approach at removing Russell's Paradox is not *totally* satisfactory — since it cannot explain just *why* the original question is not totally meaningful — nevertheless the present approach is at least *approximately* satisfactory. Thus in keeping with its own criteria, it at least *approximately* validates itself. (This is more than can be said for Popperian falsifiability, at least when taken in absolute terms!)

Russell's own response to his Paradox is contained in his Theory of Types, mentioned earlier. Actually, in its details, Russell's Type Theory admits of two versions, the "simple theory" and the "ramified theory" (the difference between which is not relevant here). But both versions have been criticised for being too *ad hoc* to eliminate the Paradox entirely.

Hitherto, other responses to the Russell's Paradox have included those of David Hilbert and the "formalists" (whose basic idea was to allow the use of only finite, well-defined and constructible objects, together with rules of inference which were deemed to be absolutely certain), and Luitzen Brouwer and the "intuitionists" (whose basic idea was that one cannot assert the existence of a mathematical object unless one can also indicate how to go about constructing it).

Yet a fourth response to Russell’s Paradox was Ernst Zermelo’s 1908 axiomatisation of set theory. Zermelo’s axioms were designed to resolve Russell’s Paradox by restricting what is called “Cantor’s naive comprehension principle”, namely that any predicate expression P(x), containing x as a free variable, will determine a set. The set’s members will be exactly those objects which satisfy P(x), namely all x’s which are P. In the case of the set of all objects that are in the room, for example, the members of that set must be all objects that are in the room: which sounds reasonable enough until we remember that some objects that are in the room can also be partly out of it, and therefore partly not in it. Thus it is now generally agreed that such an axiom must either be abandoned or modified. “ZF”, as the axiomatisation generally used today is often referred to, is a modification of Zermelo’s theory developed primarily by Abraham Fraenkel.

Nevertheless, none of the above approaches are *completely* satisfactory. But, and this is a big “but”, note that all of them are *approximately* satisfactory, even though not fully so. (And I venture to suggest that the present approach, even though it might be classed as approximately satisfactory too and not irrefutably so, it is closer to the irrefutably satisfactory than any of the others!)

In any case, the present approach validates not only itself, but also validates other approaches. It does not disqualify any attempt, providing it is at least partly successful: in other words, everyone, as it were, gets an “A” for effort, even though no one might get an “A+” for success.

SOLUTION TO THE PROBLEMS OF BUDDHIST “TRUTH”

It is noteworthy that the present approach also helps solve contradictions in other fields of thought, not only in logic. One such contradiction, which comes about especially in some forms of Buddhism, is a result of an attempt to divide truth into two kinds, “relative truth” and “absolute truth” (in Sanskrit, *samvritti satya* and *paramârtha satya*). As was pointed out in a pervious essay entitled *What Is Truth?*, the meanings of these two terms when contrasted with one another is taken to be, basically, as follows:

> The *samvritti satya* or “relative truth” (which is also at times translated as “everyday truth” or “practical truth”) is something like the truth of the statement “I paid about $20,000 for my Honda Civic”. In practical terms, if that is indeed more or less what was paid, then this statement becomes a *samvritti satya*, an “everyday (or practical) sort of truth”.
>
> However, the *paramârtha satya* or “ultimate (or highest or spiritual) truth” is that neither “I” not the “Honda Civic” nor the “$20,000” can ever be found existing independently — for after all, what can ever be found existing independently of the mind, since in order to find anything at all, mind is unquestionably required? — and so in the final analysis, and taken as an ultimate truth, nobody *really* paid any money for anything.

However, as also mentioned in my earlier essay, mixing up the two kinds of truth in a single discussion doesn't always work too well. It would be ludicrous, when caught robbing a bank, to mount a legal defence based on the ultimate non-existence of the bank, of the money and even of the robber — what to speak of the judge and the jury! Under the Buddhist division of truth into only two kinds, in a courtroom only one kind of truth, namely the "everyday" or "practical" truth, could be admitted, and the ultimate or absolute truth would have to be rejected.

However, this is obviously not satisfactory, for in the search for social justice, the higher truth is rejected in favour of the lower: much like rejecting a dollar in favour of a penny! Such an approach to justice would abase justice itself (and indeed one outcome of this approach is that Buddhism has a very poorly-worked-out concept of *social* justice, even though it does have a highly refined theory of *cosmic* justice.)

However, if instead of dividing truth into only two kinds, namely "relative truth" and "absolute truth", one divides the truth into an almost infinitely large range of truths, from the irrefutably true to the approximately true to the less approximately true all the way down to the very approximately false and even the absolutely false, then one can overcome the difficulties encountered in the Buddhist dichotomy.

One might say, for example, that what Buddhism calls *samvritti satya* is really only *approximately* true, and within truths of the *samvritti* kind, there are truths that are closer to the absolute truth or *paramârtha satya*, and others that are somewhat less close. In this way it becomes understandable for the Buddha to have enunciated his *dharma* (teaching) as an approximate truth, even while insinuating that the ultimate truth cannot really be expressed in words. Indeed it even becomes appropriate for subsequent Buddhists like Nâgârjuna and the Zen Masters to have expounded the *dharma* in their own words while still paying homage to the Buddha, and thus, at least by implication, admitting that the words of the Buddha himself must have been closer to the (ultimately inexpressible) truth than their own.

The present approach would also explain apparent contradictions in religious texts (and those, not exclusively in Buddhism). For example, Jesus is reputed to have uttered the statement "I and [my] Father are one" (*John 10:30*) as well as the statement "… my Father is greater than I" (*John 14:28*). Together these two statements, each taken as being *strictly* true, would entail a contradiction; however, if they — or even just one of them — is / are taken as approximately true, then no contradiction results. And the approximation can be as close as desired, even so close as to make no practical or for that matter discernible difference: somewhat like the value of π worked out to as many decimal places as can possibly be done. (Such an interpretation should satisfy adherents of the Doctrine of the Trinity!)

REDUCTION OF CONFLICTS BETWEEN RELIGIONS AND PHILOSOPHIES

The present approach also provides a solution to conflicts *between* religions and philosophies. Among the adherents of virtually all religions and philosophies — certainly all the major ones — there have been those who have been extraordinarily keen and brilliant thinkers; and it seems ludicrous to imagine most of them being dupes, while only the adherents of one religion or philosophy have had a monopoly on the truth (or Truth.) But that sort of thinking is an inescapable outcome of believing that a religion or philosophy can only be true or false, nothing in-between. If however degrees of approximations to the truth are admitted, then such an extreme conflict is greatly diminished. (It might still be arguable as to which religion or philosophy is *closer* to the truth than others, but such an argument need not end up becoming extreme, such as resulting in violence.)

Thus the arguments in the present essay could be used to reduce extremism between philosophical and — especially — religious viewpoints. Philosophers are not, by and large, a violent lot, but adherents of religions often are. (Actually, even philosophies have been used in modern times to justify violence, such as by the Fascists, the Nazis and the Soviets.) Such people could, perhaps, be persuaded by arguments similar to the ones given herein to take up less extreme forms of redress, such as verbal argument (as Sir Winston Churchill is reputed to have said — and it sounds better with an English accent — "To jaw-jaw is better than to war-war"!) And it is also to be noted that pretty much all people, or rather all viewpoints, have a philosophical basis: indeed a viewpoint is virtually by *definition* a philosophical stance. Thus any sane person ought to be amenable, at least partially, to the above arguments. (As for the insane, there might be other remedies, such as pharmaceutical.)

AVENUES FOR FURTHER INQUIRY

One of the avenues of further inquiry seems to be in the solution of other paradoxes. Take for instance what is called "the Berry Paradox", which was first suggested by an Oxford University librarian, Mr G.G. Berry, to Bertrand Russell (who first published it.) As explained by Dr G.J. Chaitin of the IBM Research Division, in an article published at <http://www.cs.auckland.ac.nz/CDMTCS/chaitin/unm2.html>

> Here is a version of the Berry paradox:
>
> " the first positive integer that cannot be specified in less than a billion words".
>
> This is a phrase in English that specifies a particular positive integer. Which positive integer? Well, there are an infinity of positive integers, but at any given time there are only a finite number of words in English. Therefore, if you have a billion words, there's only going to be a finite number of expressions of any given finite length. But there's an infinite number of positive integers. Therefore most positive integers require more than a billion words to describe. So let's just take the first one.

> But wait a second. By definition this integer is supposed to take a billion words to specify, but I just specified it using much less than a billion words! That's the Berry paradox.

This paradox is similar to the paradox of defining the undefinable. If the undefinable is defined as "That which cannot be defined" — which sounds reasonable enough — then we have just defined the undefinable, which sounds preposterous!

One way in which such paradoxes might be solved is to say the above definition of the undefinable is not a *perfect* definition, because if it were, it would make *perfect* sense, whereas it doesn't: it *almost* makes perfect sense but not quite. Thus one might say that the definition "That which cannot be defined" is *almost* a definition of the undefinable, but not a *perfect* definition of it; and similarly, the specification "the first positive integer that cannot be specified in less than a billion words" is *almost* a specification of that integer, but not quite a *perfect* one.

Of course one might also argue that defining the undefinable in the manner that was done above is actually defining the *term* "the undefinable", and not defining anything *real* that is undefinable. The same applies to specifying the unspecifiable (or specifying within certain parameters that which is unspecifiable within them.) Then the way out of the above paradoxes becomes one wherein the meaning of the phrase is brought under question.

Thus the approach outlined in this essay does not seem to be *absolutely* necessary to remove such paradoxes.

GÖDEL'S THEOREM

It might however be intriguing to apply the present approach to Gödel's Incompleteness Theorem. Gödel argues that if the statement:

This statement is false

… (which is what Epimenides implies by saying "All Cretans are liars") be modified to read:

This statement is unprovable

… then either it is true, in which case it *is* unprovable, which would render it true but unprovable; or else it is false, in which case it is *provable*, which in turn means it must be true, which means it can't be false. The second possibility results in a contradiction; however, the first does not. Thus only the first possibility can be true, namely that the statement must be true but unprovable.

This Edition: May 26, 2000

Note that there are only two choices given above: either the statement "This statement is unprovable" is true, or it is false. No approximations are allowed.

(Of course the above is the "crude" form of the Liar Paradox, since the word "This" in the sentence "This statement is false" attempts to refer to the entire sentence — which of course includes itself — when it obviously can't. Also, at the point at which the word "This" is just read or heard, the entire statement itself is not yet fully read or uttered, and thus the word "This" cannot possibly refer to the entire statement. But that problem is overcome by Gödel using his system of numbering, which however need not be gone into here in depth, since all it does is prove with Teutonic thoroughness that self-reference of the thought behind the sentence "This statement is unprovable" is indeed possible.)

Anyway, it is obvious that the Liar Paradox is tacitly *assumed* in order to formulate Gödel's Theorem. (This is also mentioned in the *Encyclopaedia Britannica* at its pages on *Logic*: <http://www.britannica.com/bcom/eb/article/3/0,5716,118173+16,00.html>; and also at <http://users.ox.ac.uk/~jrlucas/simplex.html>, in which the author, J.R. Lucas of the British Academy (formerly of Merton College, Oxford), has given a "A Simple Exposition of Gödel's Theorem". As he says in a related and linked article entitled *The Implications of Gödel's Theorem*, "Gödel's great achievement was to produce a water-tight version of the Epimenides Paradox".)

But if there is no Liar Paradox to begin with, or if the statement "This statement is unprovable" can have approximate truth values, then should it not be asked whether Gödel's Theorem can be proved under those conditions? Maybe it can; but then again, maybe not. It might be worth inquiring into this at some depth, especially as the implications of Gödel's Theorem have great bearing on the future of computing. (Perhaps, if it is true that "Gödel's great achievement was to produce a water-tight version of the Epimenides Paradox", then Gödel's Theorem must be based on the tacit assumption that a proposition can be either true or false, but not in-between; and if that assumption is itself shown to be false at least partially, then Gödel's Theorem might fall down!)

WHAT IS MEANT BY "PROVABLE"?

It is to be noted that one problem lies with what is meant above by the words "provable" and "unprovable". As Dr Chaitin explains in the same article mentioned earlier:

> What do we mean by "unprovable"?
>
> In order to be able to show that mathematical reasoning has limits you've got to say very precisely what the axioms and methods of reasoning are that you have in mind. In other words, you have to specify how mathematics is done with mathematical precision so that it becomes a clear-cut question. Hilbert put it this way: The rules

> should be so clear, that if somebody gives you what they claim is a proof, there is a mechanical procedure that will check whether the proof is correct or not, whether it obeys the rules or not. This proof-checking algorithm is the heart of this notion of a completely formal axiomatic system.
>
> So "This statement is unprovable" doesn't mean unprovable in a vague way. It means unprovable when you have in mind a specific formal axiomatic system {FAS} with its mechanical proof-checking algorithm. So there is a subscript:
>
> "This statement is unprovable $_{FAS}$!"
>
> And the particular formal axiomatic system that Gödel was interested in dealt with the positive integers and addition and multiplication, that was what it was about. Now what happens with "This statement is unprovable"?
>
> Remember the liar paradox:
>
> " This statement is false!"
>
> But here
>
> "This statement is unprovable $_{FAS}$!"
>
> the paradox disappears and we get a theorem. We get incompleteness, in fact.

Now under the approach used in my essay, *can* there be a proof-checking algorithm, a mechanical procedure that will check whether a proof is correct or not, whether it obeys the rules or not? Can there be a rule that might specify, for example, whether a particular experience is painful or not? To me it seems that there cannot be one.

And yet there can be no doubt that when pain exists, it does exist! Indeed if the pain is intense and prolonged enough, even the most jaded of sceptics will admit to its existence, howsoever reluctantly. What better proof could one require? It's just that there's no *mechanical* procedure for checking whether the proof is correct or not. In fact, the proof of the pain, like the proof of the pudding, is utterly non-mechanical, and altogether in the mind.

Indeed in the above-mentioned article by Dr Chaitin — in actual fact it is a transcript of a lecture given by him — he attempts to replace the Liar Paradox with Berry's Paradox and thereupon get a somewhat different outcome than was obtained by Gödel. Dr Chaitin apparently sent a paper of his viewpoint to Prof. Gödel, and even fixed with Prof. Gödel an appointment to discuss it, but unfortunately the interview never took place.

The question, at all events, is what happens if the tacit assumption of the Liar Paradox is at least partially abandoned in the formulating of Gödel's Theorem. Would Gödel's Theorem still hold true? Worth asking, what?

CONCLUSION

In conclusion, it is to be noted that the present approach is powerful enough to eliminate a large number of paradoxes in logic, though not all. It might also be suitable as a philosophical approach to solving many "real life" problems, not only in such fields as engineering (in which fuzzy logic has already proved its worth), but in the social sciences, religion and philosophy as well. It explains, for example, why science, even though it might not be able to enunciate *strictly* true positive empirical statements, is still capable of having a huge impact on the real world. (This cannot be explained by strict Popperian falsifiability.)

In addition, the present approach can also reconcile differing religious viewpoints, at least partly if not fully, and if it were adopted in negotiations aimed at promoting peace, might prevent much bloodshed. Also, although it has not yet been mentioned by me above, the present approach might help in such things as the administration of justice. For instance, if instead of returning a verdict of only "guilty" or "not guilty", a jury might be permitted to return verdicts which varied in degrees of guilt, it might serve to better fit the punishment to the crime. (This seems to be already happening in a small way: for instance, the Swiss justice system has three verdicts available: "Guilty", "Not Guilty" and "Not Guilty by Reason of Doubt". This third verdict takes care of those cases when there is not enough evidence to convict beyond reasonable doubt, but still too much evidence for the accused to go scot-free and yet for that result to be compatible with the notion of fundamental justice. As a result, a person against whom such a verdict is pronounced is released with a stain on his character, even though he does no jail time.)

Perhaps most importantly, the present approach to logic fits in with "the real world" much better than any other kind of logic: or at least any other kind of logic that I know of. A statement attributed to Marvin Minsky, the celebrated MIT Artificial Intelligence expert, goes: "Logic does not apply to the real world". And a statement attributed to von Neumann, one of the greatest mathematicians of the twentieth century, goes: "Pure mathematics is easy compared to the real world!" And there's this from Morris Kline: "Logic is the art of going wrong with confidence." If such is indeed the case, then is there not a crying need for a logic and math that *does* apply to the real world, and which *won't* lead us all marching confidently down the wrong path? The present approach seems to at least point in the right direction.

APPENDIX:

RIGOROUS TREATMENT OF THE ARGUMENTS IN THE PREVIOUS ARTICLE

This Argument Finalised on: Friday, May 26, 2000

1. Assume Two-valued Logic:

Let any proposition **p** admit of two (and *only* two) truth-values[i]:

1. 100% (or 1.0) true, or

2. 0% (or 0.0) true (*i.e.*, false).

Semantically, the meaning of this is that the membership of the proposition **p** in the set of all propositions that are true is either 1.0 or 0.0 — in other words, that **p** *is* a member of the set of all propositions that are true, or is *not* a member of that set. Absolutely *no* other possibilities are allowed.

In the notation of symbolic logic, this would read:

$$\mathbf{(q \vee \sim q)}$$

or …

$$\mathbf{(q \vee \sim q) \wedge \sim(q \wedge \sim q)}$$

[Here, the symbols "∨" and "∧" stand for the "or" and "and" operators, respectively, and the symbol "~" for the "not" operator.]

The above are, of course, the axioms of two-valued logic.

Now, using the notation of Prof. Karlis Podnieks, Dr. Math., University of Latvia, Institute of Mathematics and Computer Sciences: e-mail podnieks@cclu.lv — see also <http://www.ltn.lv/~podnieks/gt5.html#BM5_1>) let following proposition **q** be asserted:

q: q is false

… or, in the notation of Dr Dale Myers of the University of Hawaii, Dept. of Mathematics: dale@math.hawaii.edu, writing in the "Math Insight Project" (see <http://www.math.hawaii.edu/~dale/godel/godel.html>):

q iff q is false

… or, in the notation of symbolic logic,

q ≡ ~q

Notes:

[1] For the purposes of the following argument the symbol "**:**" means "such that";

[2] the term "**iff**" means "if and only if", and for the purposes of the following argument may be considered semantically equivalent to"**:**" meaning "such that";

[3] the symbol "≡" means "is materially equivalent to", and for the purposes of the following argument may be considered semantically equivalent to "**iff**";

[2] Tarski's *Self-Reference Lemma* — which for the purposes of the following argument may be accepted as being both true and satisfactorily proven — states that in adequate mathematical theories, such equations as

q: q is false

… always have solutions.

Now this proposition

q ≡ ~q

… is the same as saying:

q: q is false (*i.e.*, q is 0.0 true)

or …

q iff q is false (*i.e.*, q is 0.0 true)

or …

~(q ≡ ~q)

And *this* is the same as saying:

q: q is false (*i.e.*, q is 0.0 true) but q is not 1.0 true

or …

q iff q is false (*i.e.*, q is 0.0 true) but q is not 1.0 true

or …

$$\mathbf{q} \equiv \sim\mathbf{q} \wedge \sim\mathbf{q}$$

(*Note:* Since there is no symbolic notation for the natural language term "but", for the purposes of the following argument it may be considered semantically equivalent to the logical operator "and", namely "∧".)

In consequence of the above statements, if two-valued logic is assumed, a paradox results, since:

(1) If **q** is (100%, or 1.0) true, then **q** is *not* 0.0 true. A contradiction results.

(2) If **q** is 0.0 true, then it asserts an absolute truth (*i.e.*, it can be inferred that **q** *must* be 1.0 true). Again, a contradiction results.

Since there are *no* other possibilities, a paradox results due to the above two contradictions.

Indeed for this reason, in two-valued logic the term

$$\mathbf{q} \equiv \sim\mathbf{q}$$

… is not allowed.[ii]

2. Assume Three-valued Logic:

Let any proposition **p** admit of three (and *only* three) truth-values:

(3) 100% (or 1.0) true, or

(4) 0% (or 0.0) true (*i.e.*, false) or

(5) 50% (or 0.5 or 1/2) true (or this "indeterminate".)

Semantically, the meaning of this is that the membership of the proposition **p** in the set of all propositions that are true is either 1.0, or 0.0, or both 1.0 and 0.0 to the degree of 50% membership in each — or in other words, that **p** *is* a member of the set of all propositions that are true, or is *not* a member of that set, or is *both* a member and *not* a member of that set. No other possibilities besides these three are allowed.

In symbols, these axioms can be expressed as:

$$(\mathbf{q} \vee \sim\mathbf{q} \vee \boldsymbol{i}\mathbf{q})$$

… where the symbol "***i***" stands for "indeterminate".

Now if we assert the following proposition **q** (and for the sake of brevity we shall henceforth dispense with the "**iff**" notation):

q: q is false

or …

$$\mathbf{q} \equiv \mathbf{\sim q}$$

… then this does *not* result in a paradox, for if **q** is indeterminate, or has a truth-value of (1/2) or 0.5, then **q** can be *both* false *and* true. Thus assuming that **q** is indeterminate, then the following relation holds:

q: q is both true and false

or …

$$\mathbf{q} \equiv \boldsymbol{i}\mathbf{q} \equiv (\mathbf{q} \vee \mathbf{\sim q})$$

… which is to say,

q: q is indeterminate

or, in another notation,

$$\mathbf{q} \equiv \mathbf{0.5q}$$

or, in yet another notation,

$$\mathbf{q} \equiv \mathbf{(1/2)q}$$

Under such circumstances, the proposition

q: q is false

or …

$$\mathbf{q} \equiv \mathbf{\sim q}$$

is itself only half true (or indeterminate), which is to say,

$$\boldsymbol{i}(\mathbf{q} \equiv \mathbf{\sim q})$$

As a result of which:

$$\boldsymbol{i}\mathbf{q} \equiv \boldsymbol{i}\mathbf{q}$$

[Note that under three-valued logic, $\boldsymbol{i}(\sim \mathbf{q}) \equiv \boldsymbol{i}\mathbf{q}$**].**

As a result, no contradiction ensues, and thus no paradox results.

BUT if the following proposition is asserted:

> **q: q is false — (*i.e.,* q is 0.0 true) — or q is 0.5 true (*i.e.,* q is indeterminate).**

… which is the same as saying:

> **q: q is false — (*i.e.,* q is 0.0 true) — or q is 0.5 true (*i.e.,* q is indeterminate), but q is not 1.0 true.**

or …

$$\mathbf{q} \equiv \sim\mathbf{q} \vee \boldsymbol{i}\mathbf{q}$$

or …

$$\mathbf{q} \equiv (\sim\mathbf{q} \vee \boldsymbol{i}\mathbf{q}) \wedge \sim\mathbf{q}$$

In this case, a paradox *does* result, since, upon opening out the term above, we get two terms separated by the ∨ ("or") operator:

$$(\mathbf{q} \equiv \sim\mathbf{q} \wedge \sim\mathbf{q}) \vee (\mathbf{q} \equiv \boldsymbol{i}\mathbf{q} \wedge \sim\mathbf{q})$$

… both of which terms (*i.e.,* the ones to the right and to the left of the ∨ operator) are disallowed by the axioms and rules of inference of the above-defined three-valued logic.

Or, in plain language:

(6) If **q** is (100%, or 1.0) true, then **q** is neither 0.0 true nor 0.5 true. A contradiction results.

(7) If **q** is 0.0 true, then by inference it asserts an absolute truth (*i.e.,* **q** is 1.0 true). Again, a contradiction results.

(8) If **q** is 0.5 true, then too by inference it asserts an absolute truth (*i.e.,* **q** is 1.0 true). Once again, a contradiction results.

Since there are *no* other possibilities, a paradox results due to the above three contradictions.

The same sort of argument can be extended to four-valued logic, five-valued logic, six-valued logic, … ***n***-valued logic (where ***n*** is any finite integer greater than two).

For example,

3. Assume Five-Valued Logic

Let any proposition **p** admit of five (and *only* five) truth-values:

(1) A truth-value of 1.0 (*i.e.,* totally or absolutely true), or

(2) A truth-vale of 0.0 (*i.e.,* totally or absolutely false) or

(3) 0.*a* true (where *a* is any finite integer greater than zero),

(4) 0.[*a*+*b*] true (where *b* is likewise any finite integer greater than zero), or

(5) 0.[*a*+*b*+*c*] true (where *c* is, again likewise, any finite integer greater than zero).

Now assert the following proposition **q**:

> **q: q is false — (*i.e.,* q is 0.0 true) — or q is 0.*a* true, or q is 0.[*a*+*b*] true or q is 0.[*a*+*b*+*c*] true.**

This is the same as saying:

> **q: q is false — (*i.e.,* q is 0.0 true) — or q is 0.*a* true, or q is 0.[*a*+*b*] true or q is 0.[*a*+*b*+*c*] true, but q is not 1.0 true.**

To put it in symbolic notation:

$$\mathbf{q} \equiv \sim\mathbf{q} \vee (0.a)\mathbf{q} \vee (0.[a+b])\,\mathbf{q} \vee (0.[a+b+c])\mathbf{q}$$

or …

$$\mathbf{q} \equiv (\sim\mathbf{q} \vee (0.a)\mathbf{q} \vee (0.[a+b])\,\mathbf{q} \vee (0.[a+b+c])\mathbf{q}) \wedge \sim\mathbf{q}$$

A paradox results, since upon opening out the above terms we get:

$$(\mathbf{q} \equiv \sim\mathbf{q} \wedge \sim\mathbf{q}) \vee (\mathbf{q} \equiv ((0.a)\mathbf{q} \wedge \sim\mathbf{q}) \vee (\mathbf{q} \equiv ((0.[a+b])\mathbf{q} \wedge \sim\mathbf{q}) \vee$$
$$(\mathbf{q} \equiv ((0.[a+b+c])\mathbf{q}) \wedge \sim\mathbf{q})$$

It many be noted that *all* the terms separated by ∨ ("or") operators are disallowed by the axioms and rules of inference of the above-defined five-valued logic. And since in five-valued logic no *other* terms are allowed, a paradox does result.

Or, in plain language,

(1) If **q** has a truth-value of 1.0 (*i.e.*, **q** is totally or absolutely true), then **q** is neither 0.0 true nor 0.*a* true nor 0.[*a*+*b*] true nor 0.[*a*+*b*+*c*] true. A contradiction results.

(2) If **q** is 0.0 true (*i.e.*, **q** is totally or absolutely false) then by inference it asserts an absolute truth (*i.e.*, **q** is 1.0 true). Again, a contradiction results.

(3) If **q** is 0.*a* true, then too by inference it asserts an absolute truth (*i.e.*, **q** is 1.0 true). Once again, a contradiction results.

(4) If **q** is 0.[*a*+*b*] true, then too by inference it asserts an absolute truth (*i.e.*, **q** is 1.0 true). Once again, a contradiction results.

(5) If **q** is 0.[*a*+*b*+*c*] true, then too by inference it asserts an absolute truth (*i.e.*, **q** is 1.0 true). Once again, a contradiction results.

Since there are *no* other possibilities, a paradox results due to the above five contradictions.

The same sort of result may be obtained for ***n***-valued logic, if ***n*** is any finite integer greater than 2. (*Note:* there cannot be a *one*-valued logic, let alone a *zero*-valued logic!)

HOWEVER:

4. Assume a Logic where the Number of Truth Values is Unbounded:

Let a proposition **q** admit of truth-values whose total number is *unbounded* — which is to say, the number of truth-values the proposition **q** can admit of is not limited to any *pre*-determined number ***n***.

This means that if there is a pre-determined number ***n***, any proposition **p** may bear truth-values as follows:

(1) 100% (or 1.0) true, which is to say totally or absolutely true,

(2) 0.999…9 (to ***n*** decimal places) true,

(3) 0.999…8 (to ***n*** decimal places) true

(4) …

(10^n-1) 0.000…1 (again to ***n*** decimal places) true,

(10^n) 0.000…099…9 (now to ***n+1*** decimal places) true,

(10^n+1) 0.000…099…8 (again to ***n+1*** decimal places) true,

(10^n+2) …

$(10^{n+m}-1)$ 0.000…000…1 (to ***n+m-1*** decimal places) true, and

(10^{n+m}) 0.0 true (*i.e.*, totally or absolutely false.)

In this respect, the following definition applies:

> **To say of a proposition p that it is "0.*x* true", where *x* is any finite integer greater than zero, means that it neither 100% (or 1.0) true nor 0% (or 0.0) true — *viz.*, false, but somewhere in-between: its exact position in between the values 1.0 and 0.0 being exactly 0.*x* (whichever integer *x* may be); and as a result, its degree of membership in the set of all propositions that are true is 0.*x*.**

Thus:

> **To say of a proposition p that it is 0.*x* true means that it belongs to the set of all propositions that are totally true by a degree of 0.*x*, and to the set of all propositions that are totally false by a degree of 0.(1-*x*).**

Now bearing in mind the pre-determined number ***n***, assert the following proposition **q**:

> **q: q is 0.0 true (i.e., totally false) or q is 0.000…1 [worked out to *n* decimal places] true or q is 0.000…2 [also worked out to *n* decimal places] or … q is 0.999…9 [once again worked out to *n* decimal places] true.**

This is the same as saying:

> **q: q is 0.0 true (i.e., totally false) or q is 0.000…1 [worked out to *n* decimal places] true or q is 0.000…2 [also worked out to *n* decimal places] or … q is 0.999…9 [once again worked out to *n* decimal places] true, but q is not 1.0 (or totally) true.**

Or, in symbolic notation:

$$\mathbf{q} \equiv (0.0)\mathbf{q} \vee (0.000\ldots1_n)\mathbf{q} \vee (0.000\ldots2_n)\mathbf{q} \vee \ldots (0.999\ldots9_n)\mathbf{q}$$

or …

$$\mathbf{q} \equiv ((0.0)\mathbf{q} \vee (0.000\ldots1_n)\mathbf{q} \vee (0.000\ldots2_n)\mathbf{q} \vee \ldots (0.999\ldots9_n)\mathbf{q}) \wedge {\sim}\mathbf{q}$$

[Here, the notation **0.*uvw...y***$_n$ — where ***u, v, w, y*** and ***n*** are each of them any digit between 0 and 9 inclusive — means the term **0.*uvw...y*** is worked out to ***n*** decimal places.]

Now:

(1) If **q** is 1.0 true (*i.e.,* totally or absolutely true), then **q** is neither 0.000…1 true nor 0.000…2 true nor … 0.999…9 true. If so, q is *not* 1.0 true. A contradiction results.

(2) If **q** is 0.000…1 true, then by inference **q** is also 1.0 true: which, according to the (final) assertion of **q** itself, it is not. A contradiction results.

(3) If **q** is 0.000…2 true, then by inference again, **q** is also 1.0 true. Again, a contradiction results.

(4) If **q** is 0.000…3 true, then by inference once again **q** is also 1.0 true. Once again, a contradiction results.

(5) …

(10^n) If **q** is 0.999…9 true, then once again **q** is also 1.0 true. Once again, a contradiction results, though just barely.

BUT note that here, the proposition q *can* take on *yet another* truth-value, one that is *not* on the above list! Thus for example:

(10^n+1) If **q** is — say — 0.000…01 true (worked out to ***n+1*** decimal places), then **q** is not *totally* or *absolutely* false (*i.e.,* **q** is not 0.0 true), but then, neither is it *totally* or *absolutely* true (*i.e.*, **q** is not 1.0 true.) No contradiction results, and thus no paradox.

In standard symbolic notation modified for logic in which truth-values can be unbounded, this becomes:

$$\mathbf{q \equiv (0.0)q \lor (0.000\ldots1_{\mathit{n}})q \lor (0.000\ldots2_{\mathit{n}})q \lor \ldots (0.999\ldots9_{\mathit{n}})q \;\land (0.000\ldots01_{\mathit{n+1}})q}$$

Since there can always be a truth value of **q** worked out to ***n*** decimal places, where both ***n*** and ***m*** are finite integers greater than zero, the truth value of **q** can be anything between 1.0 and 0.0 *exclusive* of 1.0 and 0.0, *and* worked out to (***n+m***) decimal places!

Thus, for example, if *x* is any finite integer greater than zero,

(10^n+*x*) **q** is **0.*uvw…yz*** true (where ***u, v, w, y*** and ***z*** are each of them any digit between 0 and 9 inclusive, and the whole expression **0.*uvw…yz*** is worked out to ***n+1*** decimal places), in which case **q** can be *either* or *neither* 0.0 true (i.e., false) *or* or *nor* 0.000…1 [worked out to ***n*** decimal places] true *or* or *nor* 0.000…2 [also worked out to ***n*** decimal places] *or* or *nor*… 0.999…9 [once again worked out to

n decimal places] true, and *yet* be *neither* 1.0 true (*i.e.*, absolutely true) *nor* 0.0 true (*i.e.*, absolutely false)!

Thus no contradiction ensues; and thus, again, no paradox results.

A Simple Contradiction is Not a Paradox

It should be noted that a simple contradiction — or even a long series of contradictions — does not by *itself* constitute a paradox. A paradox *only* results if, given any particular set of parameters, nothing *but* contradictions result. (It will have been noticed that this has been indicated on pages 30 and 31 above: under three-valued logic, for example, if there are only two outcomes, a paradox does *not* result.)

Thus it is necessary to show that within the given parameters, *all* possible outcomes to the problem being considered *do* result in contradictions. And as a consequence, the number of outcomes to that problem must not only be *denumerable*, but also *bounded* by a given (not just a *finite*, but a *given*) number: a number, in other words, which equals the number of outcomes possible within the given parameters. (Since the parameters are given, so too must this number be: for example, two-valued logic must have two outcomes, three-valued logic, three outcomes, … **n**-valued logic, **n** outcomes.)

If *all* the outcomes are not exhausted, there *might* be an outcome which does *not* result in a contradiction — in which case there would *not* be a paradox! After all, a paradox can only be validly called a paradox if it can be *established* that it is one: namely, by examining each and every possible outcome, and showing that they all result in contradictions, without a *single* exception.

The above three paragraphs, in a nutshell, constitute the crux of the argument on which this Essay is based.

Some Objections Anticipated and Refuted

(1) It may be objected that if **q** has a truth-value that is *not* on the list that follows the terms "**q: q is**"… , namely:

> **0.0 true (i.e., false), or**
> **0.000…1 [worked out to *n* decimal places] true, or**
> **0.000…2 [also worked out to *n* decimal places] true, or**
> **… or**
> **q is 0.999…9 [once again worked out to *n* decimal places] true**

… then **q** must be *totally* false, and cannot even be the least bit true. This is not the case *under a system of logic that admits of an* ***unbounded*** *number of truth-values,* because under a system of logic in which **q** can take on a number of truth-values that is unbounded (*i.e.*, not limited to a pre-determined number ***n***), then the proposition **q** can have a truth-value that is *not* zero *nor* 1.0, and *yet* satisfy the requirement that it has a truth-value that is not on the list, the total number of whose members is limited to the number ***n***.

Thus for example if **q** is (**$0.000\ldots05_{n+1}$**) true, then under such a system of logic it is considered to be exactly halfway between (**0.0**) true and (**$0.000\ldots1_n$**) true, but not *totally* false.

And if **q** is (**$0.000\ldots000\ldots01_{n+m}$**) true, where ***m*** is a *very, very* large integer, then under such a system of logic it is considered to be *very, very* close to (**0.0**) true and yet not *totally* false.

(2) It may be objected that the list may not be limited to the number ***n***, but may be limited to a number greater than ***n***, say (***n***+***m***). This only makes is necessary to show that under a system of logic in which the proposition **q** can take on an unbounded number of truth-values, a truth-value of **q** worked out to (***n***+***m***+***o***) decimal places (where ***o*** is yet another finite integer greater than zero) would still not be on the list. As long as the number to which the list is limited is finite, the proposition **q** can take on a yet greater number of truth-values.

(3) It may be objected that the list need not be finite, and that proposition **q**, namely

> **q: q is 0.0 true (i.e., false) or q is 0.000…1 [worked out to *n* decimal places] true or q is 0.000…2 [also worked out to *n* decimal places] or … q is 0.999…9 [once again worked out to *n* decimal places] true, but q is not 1.0 true**

… can be re-written as follows:

> **q: q has any truth-value between 0.0 and 0.999…(recurring without end) inclusive of 0.0 and 0.999…(recurring without end), but q is not 1.0 true**

… in which case a paradox *would* result.

However, this argument is not valid, for it will be seen that there is no difference whatsoever between 0.999…(recurring without end) and 1.0. To establish this conclusively, we find that difference, by subtracting 0.999…(recurring without end) from 1.0 — and the answer is 0.000…(recurring without end), between which and 0 there is no difference whatsoever.[iii]

As a consequence, the re-written proposition is the same as saying:

> **q: q has any truth-value between 0.0 and 1.0 *inclusive* of 0.0 and 1.0, but q is not 1.0 true.**

However, **q** *cannot* be re-written as above without asserting the very paradox, the existence of which, under a logic admitting of an unbounded number of truth values, *remains to be proven.* The argument “begs the question”, and is therefore logically invalid.

(4) It may be noted that the list *must* be *either* limited to a finite number or *not* limited to a finite number. There is no third choice. As a result, in either case the “Liar Paradox” can be avoided in a system of logic in which a proposition **q** can take on a number of truth-values that is unbounded (*i.e.,* not limited to any given predetermined number).

Conclusion

As a result of the above arguments, it must be concluded that under a system of logic in which a proposition **q** can take on a number of truth-values that is unbounded (*i.e.,* not limited to any given predetermined number), the “Liar Paradox” can be avoided altogether.

[i] It can be proved, using two-valued logic, that two-valued logic cannot itself be a universally valid method of reasoning. (This was recognised by Aristotle himself, the originator of two-valued logic, who admitted — perhaps reluctantly — that there *are* meaningful sentences that can be made but which cannot be either true or false: such as the sentence "There will be a sea-battle tomorrow".)

As a result, assuming the universal validity of two-valued logic as a method of reasoning results in a paradox even more damaging to two-valued logic than is the "Liar Paradox".

The argument for demonstrating this is as follows:

1. Under two-valued logic, a statement must be either true or false — *no other choices are allowed.*

2. Under two-valued logic, therefore, something either exists or it does not exist. No other choice is allowed.

3. Now as a hypothesis, assume that free will (or, synonymously, choice) does *not* exist: that there is, in other words, no possibility of *choosing* from among a number of different courses of action.

4. If free will (or choice) does not exist, then a person cannot possibly *choose* to believe one belief and reject another.

5. Thus if one person believes that free will (or choice) does *not* exist, whereas another believes that it *does*, they could never come to their respective conclusions by any sort of argument or reasoning. They must each believe what they believe simply because neither of them can have any choice in the matter.

6. As a consequence, it would be impossible to tell which of them is right.

7. And as a corollary, it would be impossible to know whether the belief that free will (or choice) does not exist is really *true.*

8. Strictly under two valued logic, if it is *not* possible to know of any belief that it is true, then it *must* be possible to know of its opposite (or, synonymously, of its negation) that it *is* true.

9. The opposite (or negation) of the belief that free will (or choice) does *not* exist is that free will (or choice) *does* exist.

10. Therefore, and again strictly under two-valued logic, it *must* be true to say that free will (or choice) *does* exist, and as a corollary, that it cannot *not* exist — which in turn proves conclusively that the assumption made earlier in No. 3 above must be false.

11. Given now that under two-valued logic, free will *must* exist, now it must also be acknowledged that any statement made about the future which entails the exercise of free will (or choice) *must* be neither true nor false, for how the future will actually turn out will depend on how the free will or choice will be exercised.

12. Thus it is rigorously proven that under strictly-applied two-valued logic, it *must* be possible to make statements that can be neither true nor false: which contradicts No. 1 above.

13. Consequently two-valued logic cannot be a universally valid method of reasoning.

14. *Q.E.D.*

[ii] It should be noted that in two-valued propositional logic, *every* conclusion can be derived from either the operators {"~" and "∨"} or {"~" and "∧"} (*i.e.*, {"not" and "or"} or {"not" and "and"}) *exclusively*. Thus the operator "≡" (*i.e.*, "materially equivalent to") can be derived from them too. One consequence of this is that the above reasoning constitutes a kind of "proof" of the Liar Paradox, although if the operator "≡" is included in the list of symbols, the Liar Paradox cannot, strictly speaking, be *proved* in two-valued symbolic logic, but is taken as an (unproved) axiom.

[iii] It may be argued that there *is* an infinitesimally small difference, greater however than zero, between 0.999 … (recurring without end) and 1.0. However, if that is admitted, then it must also be admitted that the proposition **q** can thereupon bear a truth-value that is infinitesimally even smaller and yet not zero. In either case, the paradox is removed.