CRITIQUE OF

KURT GÖDEL'S 1931 PAPER

Entitled

"ON FORMALLY UNDECIDABLE PROPOSITIONS OF PRINCIPIA MATHEMATICA AND RELATED SYSTEMS"

by

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ACKNOWLEDGEMENT

Although the present paper was authored by Ardeshir Mehta, the main ideas set forth in it originate with Ferdinand Romero. However, since English is not Mr Romero's first language, and since Mr Mehta has a good command over English, and in addition has been able to grasp the salient points of Mr Romero's critique of Gödel's 1931 paper, it has been decided that Mr Mehta should do the actual writing of the paper, with the acknowledgement that the originator of most of the main ideas expressed therein is Mr Romero. (Mr Mehta has developed some of those ideas and extended their implications.) Mr Romero's own aforementioned critique on the subject can be found on the World Wide Web at <http://www.advance.com.ar/usuarios/ferdinandro/>.

ABSTRACT

If Gödel's conclusion in his 1931 Paper entitled *On Formally Undecidable Propositions of Principia Mathematica and Related Systems* (popularly known also as "Gödel's Theorem") is assumed to be correctly proven, a contradiction can arise as follows.

Assume that what Gödel writes below *is* correct:

$\mathbf{Bew_\kappa(x) \equiv (\exists y)\ y\ B_\kappa\ x}$ (6.1) …

$\mathbf{\sim[x\ B_\kappa\ (17\ Gen\ r)] \rightarrow Bew_\kappa[Sb(r^{17}_{Z(x)})]}$ (15)

$\mathbf{x\ B_\kappa\ (17\ Gen\ r) \rightarrow Bew_\kappa[Neg\ Sb(r^{17}_{Z(x)})]}$ (16)

Hence:

1. **17 Gen r** is not κ-PROVABLE. For if that were so, there would (according to 6.1) be an **n** such that $\mathbf{n\ B_\kappa\ (17\ Gen\ r)}$. By (16) it would therefore be the case that:

$$\mathbf{Bew_\kappa[Neg\ Sb(r^{17}_{Z(n)})]}$$

while–on the other hand–from the κ-PROVABILITY of **17 Gen r** there follows also that of $\mathbf{Sb(r^{17}_{Z(n)})}$. κ would therefore be inconsistent (and, *a fortiori*, ω-inconsistent).

In that case, however, **17 Gen r** must also be not *non*-κ-PROVABLE. For if **17 Gen r** *were* non-κ-PROVABLE, then there could (according to 6.1) *not* be any **n** such that $\mathbf{n\ B_\kappa\ (17\ Gen\ r)}$. By (15) it should therefore be the case that:

$$\mathbf{Bew_\kappa[Sb(r^{17}_{Z(n)})]}$$

… which however is impossible, since if there is no **n**, there can also be no **Z(n)** — and thus, in the CLASS-SIGN **r**, the FREE VARIABLE **17** cannot be substituted by a non-existent **Z(n)**. Consequently there can be no PROPOSITIONAL FORMULA which arises as a result.[1]

Whereby Gödel's first conclusion stands contradicted.

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[1] Of course by Gödel's definition 30., $\mathbf{Sb_0(y^v{}_x) \equiv y}$; and thus it may be argued that if there is no **n**, then $\mathbf{Sb(r^{17}_{Z(n)}) \equiv r}$, and thus, in the absence of an **n**, $\mathbf{Bew_\kappa[Sb(r^{17}_{Z(n)})] \equiv Bew_\kappa(r)}$. But that argument presupposes that **r** is itself a PROPOSITIONAL FORMULA . If so, however, its GENERALISATION by means of the FREE VARIABLE **17** cannot arise — *i.e.*, there could be no **17 Gen r**. And even if, by the implications of Gödel's footnote No. 18a (*q.v.*), $\mathbf{17\ Gen\ r \equiv r}$, expression (15) ends up reading $\mathbf{\sim[x\ B_\kappa\ (r)] \rightarrow Bew_\kappa(r)}$ — or, what is the same thing, $\mathbf{\sim[Bew_\kappa(r)] \rightarrow Bew_\kappa(r)}$. In which case, by 2.18 of *Principia Mathematica*, $\mathbf{Bew_\kappa(r)}$ must be correct, contrary to Gödel's claim.

And since Gödel *needs* his first conclusion to attempt to prove his second conclusion, namely that "**Neg17 Gen r** is not κ-PROVABLE" — and thus also to attempt to prove his final conclusion, namely that "To every ω-consistent recursive class κ of FORMULAE there correspond recursive CLASS-SIGNS **r**, such that neither **v Gen r** nor **Neg (v Gen r)** belongs to **Flg(**κ**)** (where **v** is the FREE VARIABLE of **r**)" — his final conclusion cannot be proved at all.

Indeed, since Gödel himself *does* prove, in his natural language argument No. 1. above, that **17 Gen r** cannot be κ-PROVABLE either — *i.e.*, that $\mathbf{Bew}_\kappa$**(17 Gen r)** cannot hold — then *neither* $\mathbf{Bew}_\kappa$**(17 Gen r)** *nor* **~[**$\mathbf{Bew}_\kappa$**(17 Gen r)]** can hold. This is only possible if **17 Gen r** is *not definable in* κ — *i.e.*, if **17 Gen r** does not belong to what Gödel calls "the class of ω-consistent FORMULAE κ."

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INTRODUCTION

In 1931 Kurt Gödel published a paper entitled *On Formally Undecidable Propositions of Principia Mathematica and Related Systems*, in which he purported to prove, using metamathematical arguments, that in any formal system of mathematics — such as the system described in Russell's and Whitehead's *Principia Mathematica* and the axiom system for set theory expounded by Zermelo and Fraenkel (which was later extended by John von Neumann) — there are, in Gödel's own words, "relatively simple problems in the theory of ordinary whole numbers which cannot be decided from the axioms. This situation is not", continued Gödel, "due in some way to the special nature of the systems set up, but holds for a very extensive class of formal systems, including, in particular, all those arising from the addition of a finite number of axioms to the two systems mentioned, provided that thereby no false propositions … become provable."

A thorough and detailed study of Gödel's aforementioned 1931 paper — which has come to be known popularly as "Gödel's First Incompleteness Theorem" or, often, as "Gödel's Theorem" — reveals, however, that there are critical logical fallacies[2] inherent in that document. This, as we shall show, is because Gödel's reasoning is based on an underlying syncretism[3] or ambiguity, which renders logically invalid the proof proffered by Gödel.[4]

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[2] The term "fallacy" is used here in the sense of "any property of an argument which allows a false or absurd conclusion to be deduced from true premises." (It may be noted that although Tarski has shown that the concept of "truth" cannot be defined in a *formal* system of logic, Gödel's theorem is in *meta*mathematics, which is not a formal system; and thus the concept of "truth" *does* apply to it.)

[3] As defined by Webster, *syncretism* means an "Attempted union of principles or parties irreconcilably at variance with each other." In the present case, the "principles" in question are the definitions of the term '*natural number*' as defined (a) in the formal system of mathematics being discussed *by* Gödel and (b) in Gödel's own language of metamathematics which is *about* the formal system of mathematics being discussed by him.

[4] It has often been argued that there is no *absolute* standard of logical validity, since the validity of a logical argument depends on the axioms and rules of inference initially assumed to formulate the system of logic in which the argument is made: and these are, by definition, entirely arbitrary. However, although this statement is correct, in that

PART 1-A

CRITIQUE OF *SECTION 1* OF KURT GÖDEL'S PAPER: "MAIN LINES OF THE PROOF"

In ***Section 1*** of his Paper, Gödel proffers — as he expresses it — "the main lines of the proof, naturally without laying claim to exactness." The salient points of these "main lines" are, in Gödel's own words (translated from the German by B. Meltzer)[5], as follows:

> A formula of **PM** with just one free variable, and that of the type of the natural numbers (class of classes), we shall designate a **class-sign**. We think of the class-signs as being somehow arranged in a series, and denote the **n**-th one by **R(n)**; and we note that the concept "class-sign" as well as the ordering relation **R** are definable in the system **PM**. Let α be any class-sign; by **[**α**; n]** we designate that formula which is derived on replacing the free variable in the class-sign α by the sign for the natural number **n**. The three-term relation **x = [y; z]** also proves to be definable in **PM**. We now define a class **K** of natural numbers, as follows:
>
> $$n \in K \equiv \sim(\mathrm{Bew}\ [R(n); n]) \qquad (1)$$
>
> (where **Bew x** means: **x** is a provable formula). Since the concepts which appear in the definiens are all definable in **PM**, so too is the concept **K** which is constituted from them, *i.e.* there is a class-sign **S**, such that the formula **[S; n]** — interpreted as to its content — states that the natural number **n** belongs to **K**. **S**, being a class-sign, is identical with some determinate **R(q)**, *i.e.*
>
> $$S = R(q)$$
>
> holds for some determinate natural number **q**. We now show that the proposition **[R(q); q]** is undecidable in **PM**. For supposing the proposition **[R(q); q]** were provable, it would also be correct; but that means, as has been said, that q would belong to **K**, *i.e.* according to (1),

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there is no absolute standard of logical *validity*, an absolute standard of logical *non*-validity *does* exist: namely total and *absolute* contradictions! In other words, if a logical argument is capable of deriving total and absolute contradictions using the very axioms and rules of inference assumed to formulate the system of logic in which the argument is made, that argument cannot possibly be valid in *any* system of logic. As we shall see, in order to prove his Theorem, Gödel uses, by implication, certain definitions and formulae from which, using his own definitions, formulae and rules of inference, complete and absolute contradictions can be derived: and for this reason, we claim that his Theorem lacks logical validity.

[5] An HTML version of this document may be found at <http://www.ddc.net/ygg/etext/godel/godel3.htm>, and a print version may be purchased from Dover Publications Inc., New York. However, there are numerous typographical errors in this document, and it should be checked against other versions of Gödel's Paper including the original German version.

> **~(Bew [R(q); q])** would hold good, in contradiction to our initial assumption. If, on the contrary, the negation of **[R(q); q]** were provable, then **~(q ∈ K)**, *i.e.* **Bew [R(q); q]** would hold good. **[R(q); q]** would thus be provable at the same time as its negation, which again is impossible.

RULES NECESSARY FOR VALIDLY ESTABLISHING A *META*MATHEMATICAL PROOF

In order to become aware of the fallacies inherent in the above argument, it is necessary to bear in mind at all times that Gödel's purported proof is argued in the language of *meta*mathematics, not in that of mathematics (which latter is the object-language of his purported proof.)

Whenever a language is used to talk *about* a language, it becomes a metalanguage. If English is used to talk about French, then the object-language is French, while the metalanguage is English. Even if English is used to talk about English, the English being used to talk *about* English is the metalanguage, while the English which is talked about is the object-language.

Similarly, if a system of logic is used to discuss another system of logic, the first is the metalanguage of the second. Thus when Gödel uses metamathematics to talk *about* mathematics, the language he uses to establish his Theorem is a metalanguage, while the object-language of his Theorem is mathematics.

Now according to the rules necessary to conduct a logically valid argument in a metalanguage, one must — as Quine puts it —make a clear distinction between the *use* and the *mention* of terms. As explained by Howard DeLong of Trinity College, Hartford in his book *A Profile of Mathematical Logic:*

> In the object language we [may][6] use the symbols, words, sentences, etc., of the object language, but [may] not mention them; in the metalanguage we [may] use the symbols, words, sentences, etc., of the metalanguage to mention the expressions of the object language, but we [may] make no use of the expressions of the object language. Instead we [must] use the names of the symbols of the object language. The same distinction can be applied to other areas. For example, we have logic and metalogic, mathematics and metamathematics, etc.

The reason for these rules — which we shall hereinafter refer to as the "use / mention rules" for establishing a proof in a metalanguage — is of course obvious: for if they are not followed, terms end up being defined ambiguously, allowing absurd and even contradictory conclusions to be derived therefrom. For example, if one does not distinguish between the object-language term *apple* and the metalanguage term *'apple'*, one might attempt to argue as follows:

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[6] The words *[may]* and *[must]* have been added by us in order to remove all ambiguity from the above-quoted words of DeLong.

Apples and lemons ripen in the summer.
Apples and lemons are six-letter words.
Therefore at least some six-letter words ripen in the summer.

Such a conclusion, of course, would be absurd — *i.e.*, neither false nor true, but rather non-sensical. It arises, however, because the words *apples* and *lemons* have been written in the same way in both the object-language and in the metalanguage. This misleads the reader into thinking that the meaning of *apples* in the metalanguage is identical with its meaning in the object-language; and likewise for *lemons*. Such a confusion can be avoided by writing object-language terms differently from metalanguage terms.

For example, one correct way to express the above argument, which would easily reveal the fallacy inherent in it, would be something like this:

Apples and lemons ripen in the summer.
'Apples' and 'lemons' are six-letter words.
Therefore at least some six-letter words ripen in the summer.

Here it is much easier to see where the fallacy lies, for the terms *'apples'* and *'lemons'* are not the same as the terms *apples* and *lemons*: the latter are fruits, the former, words. Here we have distinguished the terms *'apples'* and *'lemons'* — the words — as defined in the metalanguage by writing them within single quote marks,[7] thereby distinguishing it from the terms *apples* and *lemons* — the fruits — which are the object-language terms, and therefore not written within any quote marks.[8]

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[7] Single quote marks are normally used so as to not be confused by actual quotations, which are normally written within double quote marks. But it is not absolutely necessary to use single quote marks: double will do just as well.

[8] The reason why the terms *'apples'*, *'lemons'*, *apples* and *lemons* are in italics above is because in those paragraphs we are using, not just a metalanguage, but a *meta*-metalanguage: for here we are talking *about* not just an object-language, but an object-language *plus* a metalanguage. *Apropos*, one serious and indeed valid objection levelled against the use / mention rules for establishing a proof in a metalanguage is that these rules themselves talk about *both* the object-language *and* the metalanguage, and thus must be in a meta-metalanguage. But those very rules must apply to the meta-metalanguage also, and thus must be part of a meta-meta-metalanguage … and so on *ad infinitum*. (This is akin to the objection that if everything is caused, there could never be a "first cause", which by definition must be uncaused, and this would imply an eternity of time in the past for all things to be caused by previous causes; whereas if everything is *not* caused, then there is no way to assert of any observed phenomenon that it *must* have had a cause, and then anything becomes possible: one could always assert that such-and-such new galaxy, only discovered today, popped into existence just yesterday.) There is no satisfactory answer to such objections, and it may well be that the human mind is incapable of dealing with the infinite — and the eternal — in a completely satisfactory manner: indeed the Intuitionist, Constructivist and Finitist schools of mathematics deny that any such thing as an infinite number can meaningfully be asserted to exist at all. It is almost inevitable that we human beings will never be able to satisfactorily answer *all* questions, especially about the infinite and the eternal, for if we could, we'd very likely be omniscient; but that does not necessarily mean that the use / mention rules for establishing a proof in a metalanguage are necessarily useless or should not be applied when called for. They happen to be the best we have at present, and until something better comes along, there is no sense in *not* applying them — just as there is no sense in not using our rockets, Space Shuttles, *etc.* for space travel until "warp drive" (as in *Star Trek*) becomes available.

It is relatively easy to see this difference between object-language and metalanguage when the meaning of the term in question differs significantly in the former as opposed to the latter. But it is more difficult to see the fallacy when the meanings are the same in both object-language and metalanguage. Take the word *word*. When English is used to talk about English, is the word *word* in the object-language the same as the word *'word'* in the metalanguage?

One may think, Yes, it is: after all, how can the English word *word* mean other than "word"? But it is *still* important to distinguish in which language the word *word* is being expressed: in the object-language or the metalanguage. This can easily be seen by considering its plural, *words*. In the metalanguage, the object-language word *'words'* is just one word, not many; whereas in the object-language, the object-language word *words* signifies a plurality of words.

This can also be seen from the following "trick question":

YOUNG IMP: Dad, when's one word actually two words?

DUMB DAD: *(After thinking a bit)* I give up. When?

YOUNG IMP: *Always!* "One word" is always *two* words — "one", and "word"!

The trick here lies in making the DUMB DAD think the question is being asked in the object-language, when all the while it is being asked in the metalanguage — as can be seen by the quote marks accompanying the words "one" and "word" in the last line.

Now it is to be noted that Gödel's Theorem takes advantage of a paradox structure in order to arrive at its conclusion. Indeed Gödel himself writes in his footnote No. 14, "Every epistemological antinomy [*i.e.*, paradox][9] can be used for a similar undecidablity proof". However, as we shall see, at least some — if not all — paradoxes are rooted in a confusion between terms in the object-language and the "same" terms in the metalanguage. The terms are not really the *same*, even though they may *appear* to be so.

It has been argued — and, indeed, cogently — that *all* paradoxes (or antinomies) must be a result of *some* sort of linguistic confusion. The basis of this argument is the observation that in real life there *are* no paradoxes. Thus any *apparent* paradox must be just that: merely apparent. The trick is to find out just how this appearance arises. Separating object-language from metalanguage is one relatively easy way to see where the fallacy lies in many paradoxes.

Take for example "Grelling's paradox", formulated in 1908 by Grelling and Leonard Nelson. It goes more or less as follows: there are some words which possess the property they express; for example the word 'short' is short — let us call these words *autological.* And let us call all other words *heterological*. For example, 'long' would be a heterological word — it is not long.

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[9] The word "antinomy" is perhaps more accurate here, since it signifies a conflict of *laws* (from the Greek, *anti-* + *nomos* "law") whereas "paradox" signifies a statement that is *seemingly* contradictory or opposed to common sense (from the Greek *para-* + *dokein* "to think", "to seem" — see *Webster*.) But for our present purposes we shall consider the two terms to be synonymous, since we argue that most if not all antinomies are, in fact, only *seemingly* paradoxical.

(So, however, would the word 'for' be, as well as the word 'which' and the word 'and' and the word 'however', and the vast majority of others: none of them are autological, and thus by definition must be heterological.)

Now the question is asked whether the word 'heterological' is itself heterological. If yes, then the word 'heterological' possesses the property it expresses, and thus is by definition autological, not heterological. If no, then the word 'heterological' does *not* possess the property it expresses, and therefore by definition it *is* heterological. So the word 'heterological' is heterological if and only if it is *not* heterological: a paradox.

Here the fallacy distinctly lies in not distinguishing between object-language and metalanguage. The word 'heterological' is not the same thing as the *label* applied to it, nor is the word 'autological' the same thing as the *label* applied to it: whether the label is identical to the word or it is not. The former — the words — are in this case parts of the object-language, while the latter — the labels applied to them — are parts of the metalanguage. In set-theory terminology, they belong to (*i.e.*, are elements of) different sets.

This is much more easily recognised if the correct terminology is used in enunciating the so-called "paradox". The word 'short' is merely *called* autological: the term 'autological', in other words, is a mere *label* applied to the word 'short'. But just because it is *called* autological does not *make* it autological: just as a man *called* "Green" is not *himself* green in colour. The word "heterological', if it happens to possess the property it expresses, would merely be *called* (or *labelled*) "autological"; but that would not necessarily mean it *is* autological — *i.e.*, it would not possess the *property* of being autological. There is no more a paradox in this than there is in the fact that a singer called "Barry White" happens to be black.

Take as another example the Jourdain Truth-Value Paradox, which is essentially a more elaborate version of the Liar Paradox. (The Liar Paradox itself is rather easily disposed of when we realise that the "Liar sentence" — one which runs: "This sentence is false", and which thereupon asserts its own falsehood — talks *about* itself, and thus belongs to both the object-language *and* metalanguage. As a consequence, according to the use / mention rules of soundly establishing a proof in a metalanguage, it is not permissible for it to talk about itself in order to assert its own falsehood.) But suppose we have a piece of paper on one side of which is written "The sentence on the other side of this paper is true", while on the other side is written "The sentence on the other side of this paper is false." For convenience, let us call the first sentence ***A*** and the second sentence ***B***. Now if ***A*** is true, then ***B*** is false, and if ***B*** is false then ***A*** must also be false. But if ***A*** is false, then ***B*** must be true, and if ***B*** is true, then ***A*** must be true. Neither sentence talks about itself, but taken together they keep changing the truth-value of the other, so that eventually, we are unable to say whether either sentence is true or false.

The fallacy here lies in the fact that initially, ***A*** talks *about* ***B***, and it is only *then* that ***B*** talks about ***A***. As we saw, when one sentence talks *about* another, the one talked about belongs to the object-language, while the one doing the "talking" (as it were) belongs to the metalanguage. Thus ***B*** cannot be in the metalanguage of ***A*** — it is ***A*** that is in the metalanguage of ***B***. And as we saw from the quote given above from DeLong's book *A Profile of Mathematical Logic*, according to the use / mention rules for soundly establishing a proof from a metalanguistic viewpoint, in the object-language we may neither mention nor use the terms of the metalanguage. Thus it is not permissible, once ***A*** has been seen to be in the metalanguage of ***B***, for ***B*** to talk about ***A***.

To give yet another example, the Berry Paradox — so called after an Oxford University librarian, one Mr G. G. Berry, who first brought it to the attention of Bertrand Russell — is also rather easily resolved if a distinction is made between object-language and metalanguage.

A somewhat loose but relatively easily-analysed version of the Berry Paradox can be given as the following conversation between two people (say, a MR ELDER and a MR BERRY):

MR ELDER: How can you define the undefinable? I'd say that would be *quite* impossible.

MR BERRY: Oh, that's easy! I'd define the undefinable as "That which cannot be defined".[10]

The fallacy is immediately revealed by noting that MR ELDER has not *actually* defined the undefinable, in the sense of defining something that is *itself* undefinable; he has merely defined the *phrase* 'the undefinable', which *can* be defined — and, indeed, defined quite effortlessly, at that.

This would have been noticed even more easily had the "paradox" been written as follows:

MR ELDER: How can you define the undefinable? I'd say that would be *quite* impossible.

MR BERRY: Oh, that's easy! I'd define 'the undefinable' as "That which cannot be defined".

As we can see, when the "paradox" is written thus, MR BERRY is not actually answering the question asked by MR ELDER. He is answering an entirely different question, namely: "How can you define 'the undefinable'?" The original question is asked in the object-language while the answer is given in the metalanguage. The undefinable — the *real* undefinable, the one in the object-language — remains just as undefinable as ever, even after MR BERRY'S glib reposte. There is in reality no paradox at all, even though there *appears* to be one.

What Gödel tries to do, essentially, is something similar: *viz.,* to prove the unprovable.[11] And, since he does *not* distinguish between object-language terms and metalanguage terms, it *appears*

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[10] A more exact version of the Berry Paradox can be illustrated as follows: The number of syllables in the English names of finite integers tend to increase as the integers grow larger, and gradually increase indefinitely. Hence the names of some integers must consist of at least nineteen syllables, and among them there must be one which is the least of the lot. Hence the term "the least integer not definable in fewer than nineteen syllables" must denote a definite integer: in fact, it denotes the integer 111,777. But "the least integer not definable in fewer than nineteen syllables" is itself a term consisting of *eighteen* syllables; hence the least integer not definable in fewer than nineteen syllables can in actual fact be defined — as above — in *eighteen* syllables. (The term "definable" may be replaced by "nameable" or "denotable" or "specifiable" or some such equivalent term.) The fallacy here lies in using the term "definable" — or "nameable" or "denotable" or "specifiable" or equivalent, as the case may be — ambiguously, *i.e.*, in two different ways: in the first way, it means using a definition (or name or specification) that in actual fact *is* the number being defined, expressed in syllables; while in the second way, it merely *refers to* or *mentions* that number. Thus the first is a definition in the object-language, the second in the metalanguage.

that he can actually prove the unprovable — just as under those conditions it *appears* that MR BERRY can actually define the undefinable.

But proving the unprovable in any *real* sense is impossible: because if Gödel actually *succeeds* in proving that there exists a proposition which is true but is nevertheless unprovable, then obviously it couldn't *really* have been unprovable … now could it have. The first step in the reasoning — *viz*., proving that the proposition in question is true — *ipso facto* proves the proposition itself; and if it has been proved, how can it be said to be *un*provable?

Now as we shall see, a study of Gödel's aforementioned 1931 Paper reveals that nowhere in it does Gödel draw any clear distinction between the symbolism of the object-language on the one hand, and the symbolism of his metalanguage on the other.

If Gödel *were* to make a distinction between object-language and metalanguage, and were to try to show that although a particular formula is unprovable *in the object-language*, it *is* provable *in the metalanguage*, then it may be imagined that he might have *some* chance of success. (Not much, of course, because to show that, he must first establish an object-language and a metalanguage in which a formula which is *part* of the object-language is nevertheless not *provable* in the object-language, and in addition, that same formula must also both be a *part* of the metalanguage *and* be provable in it as well — all of which constitutes quite a tall order.)

But the way Gödel has *actually* enunciated his "proof" — *i.e.*, without a clear and unambiguous distinction between an object-language and a metalanguage being made — his "proof" falls flat: the unprovable remains just as unprovable as ever, even though it *appears* to be proved. That is because a proposition that asserts its *own* unprovability — and does so in only one language, *without* any distinction being made between object-language and metalanguage — must belong to both the object-language as well as the metalanguage. And as we have seen from the example of the statement "Apples and lemons are six-letter words that ripen in the summer", such a statement can be neither true nor false, but is nonsensical: a "non-proposition" in the guise of a proposition.

We can easily see how rather obvious nonsensical statements can be turned into statements that *appear* to make sense but nevertheless are still nonsensical. An example of a rather obvious nonsensical statement — a "non-proposition" — is the well-known Noam Chomsky sentence "Colourless green ideas sleep furiously". Do they or do they not? *I.e.*, is this a true or a false proposition? Obviously it is neither true nor false, because it is not a proposition to begin with — *i.e.*, it is not a sentence that can be either true or false — but rather nonsensical.

Suppose now that we were to substitute the words "Santa Claus and his elves" for the phrase "Colourless green ideas" above. Then we get the sentence "Santa Claus and his elves sleep furiously". Is *this* a meaningful sentence or not? It is certainly closer to being meaningful, but still not quite there: "sleeping furiously" is not something that can meaningfully be done.

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[11] *I.e.*, in the sense of proving that in any system of logic capable of formalising simple arithmetic, a proposition which must be true but is nevertheless unprovable has got to exist. (This, by the way, does not mean that Gödel proves that *false* propositions are unprovable. That would be a trivial attainment — *all* false propositions are unprovable, for if they weren't, they wouldn't be false!)

But now suppose we were to substitute "sleep peacefully" for "sleep furiously". Then we get the sentence "Santa Claus and his elves sleep peacefully". *Now* do we have a meaningful proposition, one which can be either true or false? Most people might say, "Yes." But that would also be a mistake, for neither Santa nor his elves have any existence, and non-existent beings can neither sleep nor be awake, whether furiously or peacefully! Thus even the sentence "Santa Claus and his elves sleep peacefully" is not really a proposition, though it *appears* to be one.[12]

Similarly, a proposition that asserts its *own* unprovability is also not a genuine proposition, a statement which can be either true or false. It merely *appears* to be so. But in reality it is as nonsensical as saying "Apples and lemons are six-letter words that ripen in the summer". That's because it does not distinguish between the object-language — in which the sentence is enunciated — and the metalanguage, in which it asserts its own provability.

As we noted earlier, one may use any method of distinguishing object-language terms from those of the metalanguage — for instance, one may colour-code the terms: using, for example, red type for metalanguage terms and black type for object-language terms. But whatever the method used — or even if it is not used at all in writing — at least *mentally* distinguishing object-language terms from metalanguage terms is absolutely necessary for establishing a proof in a metalanguage, for otherwise there is no way to be sure of the soundness of the argument.

Since as we have seen above, it is absolutely necessary for establishing the soundness of his purported metalanguage or metamathematical proof to at least mentally draw such a distinction, we shall attempt to do so hereunder, by analysing Gödel's above-quoted passage term by term and sentence by sentence, and inferring from his words what the definition of each term must be, and in which language: object-language or metalanguage.

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[12] It has been argued that although non-existent beings cannot actually *do* anything, they *can* at least *not* do something; and thus although Santa and his elves might never be able to sleep peacefully, it can at least be truthfully said that they do *not* sleep peacefully. (Russell and Whitehead make this argument in their Introduction to *Principia Mathematica*, at least by implication, when they say that the sentence "The King of France is bald" is false because there is no such thing as the King of France.) By such an argument, the sentence "Santa Claus and his elves sleep peacefully" *is* a proposition, because although it can never be true, it can at least be false. But this argument can lead to an absurdity, for if it is accepted, then the sentence "Colourless green ideas sleep furiously" would also be false rather than nonsensical! Similarly, the sentence "The square root of Richard Nixon is 1.4142135623731…" would also be a proposition, for it could also be argued that this sentence is false rather than meaningless. If the argument were accepted, in other words, there would be precious few sentences that are *not* nonsensical or meaningless, which in turn would conflict with our understanding of the term "nonsensical" or "meaningless" — indeed, we'd have to redefine the meaning of the word "meaning". (The problem arises because the argument relies on a subtle but nevertheless crucial distinction between the way the word "not" is used. In ordinary parlance, when we say a person is not sleeping, we mean he or she is awake; but under the above argument, it does *not* mean that he or she is awake — nor, indeed, does it mean that he or she is *not* awake. The same applies to the opposites "sane" *vs.* "insane", "possible" *vs.* "impossible", "clothed" *vs.* "naked", "something" *vs.* "nothing", and all such terms, one of which is normally understood to be the exact opposite of the other. Under the above argument, there must be a clear difference between "impossible" and "not possible", and likewise, between "sane" and "insane", between "not clothed" and "naked", and between all such pairs of words — *e.g.*, just because something is not possible would not necessarily mean it is impossible, nor would a person who is *not* capable of walking necessarily be *in*capable of walking.)

PART 1-B

DETAILED ANALYSIS OF *SECTION 1* OF KURT GÖDEL'S 1931 PAPER

The detailed analysis of ***Section 1*** of Gödel's aforementioned 1931 paper will be undertaken hereunder in a straightforward way: namely, by analysing Gödel's own words[13] of "the main lines of the proof" term by term and sentence by sentence. This, we expect, should be sufficient to lay bare the assumptions concealed in the purported proof, and reveal the fact that Gödel cannot prove his Theorem in the way he intends to prove it.

For the sake of clarity we shall refer to Gödel's metalanguage "The metalanguage **G**" (or sometimes simply "**G**" — without the quote marks, of course), while we shall refer to his object-language as "The object-language **PM**" or "The object-language **P**".

(i) Beginning with the first sentence of the above-quoted passage of Gödel's:

> A formula of **PM** with just one free variable, and that of the type of the natural numbers (class of classes), we shall designate a **class-sign**.

This sentence shows that the term **class-sign** *is defined in the metalanguage **G***, and here its definition is "A formula of **PM** with just one free variable of the type of natural numbers." In addition — and leaving no doubt as to what Gödel means by the term 'natural number' — he specifies it to be a "class of classes".

Note that the term 'natural number' as expressed here is the definition given to that term *in the object-language:* namely, in what Gödel calls "the system **PM**".[14]

No fallacy is evident in the sentence as it stands.

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[13] As far as possible in our critique we shall quote and refute Gödel's own words — in their English translation, of course — so as to leave the least possible doubt as to the interpretation of the terms and sentences thereof. (We shall not quote Gödel's footnotes unless they are relevant to what we happen to be saying, for in the main they serve merely to elucidate his text, and for the most part contain nothing of importance that is not also contained in the main portion of his text.)

[14] More accurately, it is the definition given to that term in the *mathematical interpretation of* the object-language **PM**. The terms of the formal system **PM**, being those of a *formal* system, do not by themselves carry any meaning; it is only when they are *interpreted* that they acquire meaning. However, Gödel is dealing exclusively with the *mathematical* interpretation of the formal system **PM**, and thus when we refer to a "definition in the system **PM**", we shall always mean that definition which accrues when the system **PM** is interpreted so as to derive the proofs of mathematics therefrom.

(ii) Gödel's next sentence is:

> We think of the class-signs as being somehow arranged in a series, and denote the **n**-th one by **R(n)**; …

Here it is clear that the term '**n**-th' used by Gödel refers to the *ordinal number* '**n**-th' *as it is defined in* ***PM***: and thus the term '**n**-th' is not expressly defined *in the metalanguage* ***G***. As a consequence, the concept of *the cardinal number* '**n**' *as it is defined in the object-language* ***PM*** — the one which corresponds to the ordinal number '**n**-th' as it is defined in the object-language **PM** — is also not expressly defined in the metalanguage **G**.[15] Although this is not a fallacy as such, it *is* a source of ambiguity leading to syncretism, as we shall show further on in our critique.

However — and admittedly — Gödel does define the term '**R(n)**', which is defined *in the metalanguage* ***G**:* and it is therein defined as the *designator* or *identifier* of the **n**-th class sign in a series of class-signs.

The word "designator" or "identifier" is used by us here in the sense of a symbol — or series of symbols — which *identifies* an object: the way the *name* of a person designates or identifies that person, or the way the postal address of a house designates or identifies a particular house, or the way a credit-card number identifies a particular credit-card, or the way a series of alphanumeric symbols on a licence-plate identifies the automobile to which that licence-plate is attached. As the above examples show, an identifier need not be exclusively alphabetical: it can be alphabetical, alpha-numeric or even purely numerical. Indeed it can even be in Chinese ideograms or Egyptian hieroglyphics — or for that matter, in entirely meaningless symbols.

When it is expressed numerically — *i.e.*, with the help of digits from **0** to **9** inclusive — an identifier might more accurately be called a label, or a "numerical name", rather than a "number" in the number-theoretical sense. In actual fact it is just a label or a name which happens to be written in digits instead of in alphabetical letters. It would make no whatsoever sense to perform numerical calculations with a numerical name, any more than it would to perform numerical calculations with a name like "Arthur": such as, for example, calculating the square root of "Arthur", or dividing it by 10, or substituting it for a free variable in a formula when that free variable must be, as Gödel puts it, "of the type of natural numbers (class of classes)".

An identifier is always in the metalanguage, for it is used to *mention* the object-language term which it identifies. In our earlier examples, the word *'apple'* identifies (or serves to mention) the fruit *apple*. Similarly, the name *'Arthur'* identifies (or serves to mention) the person *Arthur;* a chapter number identifies (or serves to refer to, *i.e.*, to mention) a particular chapter, a credit-card number identifies (*i.e.*, serves to refer to) a particular credit-card, and so on.

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[15] It may be argued that earlier in his paper Gödel does expressly define the basic signs of his metalanguage as 'natural numbers', for he writes: "For metamathematical purposes it is naturally immaterial what objects are taken as basic signs, and we propose to use natural numbers for them". However, this is definitely not a correct statement: it is *not* immaterial what objects are taken as basic signs. To be more precise, it is not permissible to take objects (*i.e.,* symbols, words, terms, *etc*.) belonging to the *object-language* as the basic signs of a *metalanguage*. That would be like using the terms *'apples'* and *'lemons'* to mean both fruits <u>*and*</u> words … which, as we saw, results in a fallacy.

Thus when the object-language is itself mathematics — as in the case of Gödel's Theorem — an identifier or label or "numerical name" cannot be a part of it. Such a term can only be part of the metalanguage. And it doesn't matter how the identifier is written, whether alphabetically, alpha-numerically or even entirely in digits from **0** to **9** inclusive: as long as it is a name or label rather than a number, it is akin to any other name, and therefore *not* part of the object-language.

Whenever a series of symbols is used as a name or a label, it loses any meaning it has in the object-language, and acquires another: *viz*., that of being the designator or identifier of the object or person of which it is the name. We see that clearly in the above example: the word *'apple'* is not the same as the fruit *apple.* This difference in meaning is due to the fact that the former belongs to the metalanguage, the latter to the object-language.

We also see this with the names of persons. Many people are named after their professions, like Smith, or after their appearance, like Black. But Adam Smith was not a smith, nor is Conrad Black black; and neither is Larry King a king.

Even when the name, on the one hand, and the profession or appearance (or attribute), on the other, do coincide, they are still essentially different from one another. Minnie Driver may, on occasion, be a driver, and perhaps one of her ancestors was a full-time driver; but that does not mean that the *name* 'Driver' is the same as, or identical to, the *profession* 'driver'. We recognise this instinctively by writing 'Driver' with an initial capital letter, while '*driver*' is written in all lower-case letters.

But even if both were written in exactly the same way, the two would not be the same, even though they would *look* the same. It would be just a trick played on the mind by two identical-looking strings of symbols which however belong to two different languages: just as the word *word* in the object-language is different from the word *'word'* in the metalanguage, so that "one word" *of* the object-language are in fact two words when *mentioned* in the metalanguage.

Thus it is immaterial in what kinds of symbols a name is written: whether in letters of the alphabet or in digits from **0** to **9** inclusive. A string of such symbols, when it is a name, is not the same thing as the same string of symbols when it is a number. The two may *look* the same, but they are not. The latter is in the object-language *viz.,* mathematics, while the former in the metalanguage, *viz.,* metamathematics.

(iii) Gödel now goes on to say:

> …and we note that the concept "class-sign" as well as the ordering relation **R** are definable in the system **PM**.

Now this is only partly true, not entirely.

It is true that the terms 'class-sign' and **R** are both definable in the system **PM**; however, as seen from *(ii)* above, both of them are not definable *in the same way* in both the object-language **PM** and in the metalanguage **G**. In particular, the ordering relation **R** is defined in the object-language **PM** as the relation between the **n**-th ordinal number and its corresponding cardinal number **n**; while in the metalanguage **G**, although the term **R** is not explicitly defined, it is by implication defined as the identifying relation between the **n**-th ordinal number (as defined in **PM**) and the class-sign of **PM** which is identified or designated by that ordinal number *via* the process of the "class-signs as being arranged … in a series" (as Gödel puts it.)

Thus these two definitions of **R** are *not* the same.

(iv) Gödel continues as follows:

> Let α be any class-sign; by **[α; n]** we designate that formula which is derived on replacing the free variable in the class-sign α by the sign for the natural number **n**.

Now the term "natural number" as used by Gödel here has been defined by him in *(i)* above as what he calls "class of classes". This is a concept belonging to the object-language **PM**. And as such, it is equivalent to the concept "cardinal number", which is also the same concept — "class of classes" — in the system **PM**.[16]

However, we have seen by *(ii)* above that the concept of "cardinal number" (class of classes) is *not* explicitly defined *in the metalanguage **G*** — which in turn means that the concept "natural number", when it too is defined as "class of classes", is not explicitly defined in the metalanguage **G**. Therefore the term **n** is not *explicitly* defined in the metalanguage **G** as a "natural number".

But as seen above, Gödel defines the term '**[α; n]**' in his metalanguage **G** as

> that formula which is derived on replacing the free variable in the class-sign α by the sign for the natural number **n**.

This definition *implies* that Gödel wishes to define **n** as a natural number *in the metalanguage **G***, and to *use* what he calls "the sign for the natural number **n**" when so defined *in* the object-language **PM**. But this is not permissible by the use / mention rules for establishment of a sound mathematical proof, for it creates a definitional contradiction.

Note well: Gödel *by implication* desires to define the term **n** as a "natural number" *in the metalanguage **G*** — and he requires this definition to be *exactly identical to the definition of the*

[16] Gödel uses the term "class" as the equivalent of the term "set" — as in "set theory". Note also — as a parallel — that Cantor denotes the cardinal number of a set *M* as "the general concept which, by means of our active faculty of thought, arises from the aggregate *M* when we make abstraction of the nature of its various elements *m* and of the order in which they are given" [1895-1897]. Or, as DeLong elucidates in his book *A Profile of Mathematical Logic*, "In other words, the cardinal number of a set is what *M* has in common with all sets equivalent to *M*. Cantor uses the symbol $\overline{\overline{M}}$ *[M-with-two-bars-above-it]* to signify the cardinal number of a set *M*. The double bar indicates a double abstraction: first, from the nature of their elements, second, from their order. If we have made only the first abstraction we have the concept of ordinal number of *M* for which Cantor uses the symbol $\overline{M}$ *[M-with-one-bar-above-it]*. … Strictly speaking, for ordinal numbers we should use the expressions "first", "second", "third", etc. … In short, cardinal numbers indicate *how many* members there are in a set, ordinal numbers the *position* of members in a sequence." Gödel's term "class of classes", therefore, exactly fits the definition of "cardinal number" as well as "natural number"; and thus the two terms may be considered synonymous for the purposes of his purported proof, as well as for its refutation by us here.

term **n** *as a 'natural number' in the system* ***PM***. However, that is not logically sound, for the term '**n**-th' has already been defined in the metalanguage **G** as the *numerical name* or *identifier* of a series of class-signs. Hence the term '**n**-th' cannot *also* be defined therein as the ordinal number corresponding to the cardinal (or natural) number **n** — as it is defined in the object-language **PM** — for that would result in *two* definitions for a single term in the metalanguage **G**: which in turn would mean that the metalanguage **G** would be rendered ambiguous.

In other words, it would be akin to defining *'apple'* in the metalanguage as both a word *and* as a fruit — which would allow one to say "An apple is a five-letter word that can be baked in a pie". Such a conclusion, of course, would be nonsensical.

Thus any expression in **G** using the term **n** defined as *both* a natural number *and* an identifier of a series of signs of the object-language **PM** would be syncretic — *i.e.*, it would attempt to unite two definitions irreconcilably at variance with each other. Such a procedure, of course, results in a fallacy — a property of an argument due to which it becomes possible to draw one or more self-contradictory conclusions, or absurd conclusions, from true — or correct — premises. (As we shall show further on, this is exactly what turns out to be the case.)

Note also that what Gödel has *actually* meant by his expression **[α; n]** is:

> That formula which is derived on replacing the free variable in the class-sign designated by the term α by the sign for the natural number **n**.

But the term α which designates a class-sign in actual fact represents one or more digits from **0** to **9** inclusive, and so does the natural number **n**!

Thus, if an example were to be given, supposing α — or **R(n)** — were to represent the **17**th class-sign in the aforementioned series, and **n** were to represent the natural number **17**, then **[α; n]** would be written **[R(17); 17]** … but the two symbols '**17**' would each represent a *different* concept. Indeed they would belong to two entirely different sets: one of them would be in the metalanguage while the other would be in the object-language.

So **[R(17); 17]** can hardly be called what Gödel intends to express by his term "meaningful".[17] It is tantamount to saying in one single sentence something like "Sugar is a five-letter word that tastes sweet." But there *is* no five-letter word that *tastes* sweet … not even the word 'sweet' itself!

And as a consequence, any definition in the metalanguage **G** of the term **n** which is at variance with the definition of the term '**n**-th' in the object-language **PM** would be what Gödel

[17] Note that Gödel writes in ***Section 1*** of his paper: "it is easy to state precisely just which series of basic signs are meaningful formulae and which are not." In modern terminology, what Gödel calls "meaningful formulae" would be called "well-formed formulae". However, although in most cases it is indeed easy to specify "which series of basic signs are meaningful formulae and which are not", in the case of Gödel's own paper it is not all that easy, until and unless all signs in all the formulae are clearly defined, and the language in which each definition is given — object-language or metalanguage — clearly specified.

would have called "not meaningful" — or what we nowadays would call "not well-formed"[18] — and ultimately result in a logically fallacious argument.

Note, however, that even if the definition of the term **n** in the metalanguage **G** were exactly the same as the definition of the term '**n**-th' in the object-language **PM**, it would still not suffice to render the **n** in the metalanguage the *same* as the **n** in the object-language. And the reason is, of course, that they belong to different languages — *i.e.*, to different collections of words, or in set-theory terminology, to different sets.

Just as we saw that it is easy to see the error in the sentence "Apples and lemons are six-letter words that ripen in the summer" by putting the metalanguage terms within single quotes, it is also easy to see the error in Gödel's formula **[R(n); n]** by putting the metalanguage '**n**' within quotes, thus: **[R('n'); n]**. Now we can easily see that **n** is not the same as '**n**'.

It may parenthetically be argued that **n** could be defined in the metalanguage **G** as an abbreviated way of writing the ordinal number '**n**-th': as is done for instance when numbering the footnotes in most documents. (To be precise — as DeLong indicates in his words quoted in one of our footnotes earlier — footnotes should be numbered "**1st**", "**2nd**", **3rd**", "**4th**", "**5th**", *etc.*; but for the sake of being concise, one often dispenses with suffixes such as "**-st**", "**-nd**", "**-rd**", and "**-th**": just as when referring to a person one often dispenses with writing the entire words "**Mister**" or "**Doctor**", and merely abbreviates them to "**Mr**." or "**Dr**." as the case may be.)

However, even that would be irreconcilably at variance with the definition of **n** required by Gödel, for *if* the term **n** is defined in the metalanguage as merely as an abbreviated way of writing the ordinal number '**n**-th', then **n** in the metalanguage **G** would be equivalent to an *ordinal* number; and as such, it could not validly replace a free variable in a formula of **PM**, if that free variable must be of what Gödel calls "the type of *natural* numbers (class of classes)".

(v) Gödel now continues as follows:

> The three-term relation **x = [y; z]** also proves to be definable in **PM**.

This too is only partly true. It *would* be true only *if* all the terms **x**, **y** and **z** belong to the system **PM**. If any of them can be demonstrated *not* to belong to the system **PM**, the above sentence

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[18] In modern terminology, a "well-formed formula" of the propositional calculus is defined thus (the following is from the *Encyclopaedia Britannica,* but any standard textbook on logic contains similar statements): "In any system of logic it is necessary to specify which sequences of symbols are to count as acceptable formulas — or, as they are usually called, well-formed formulas (wffs). … From an intuitive point of view, it is desirable that the wffs of PC be just those sequences of PC symbols that, in terms of the interpretation given above, make sense and are unambiguous; and this can be ensured by stipulating that the wffs of PC are to be all those expressions constructed in accordance with the following PC formation rules, and only these 1. A variable standing alone is a wff. 2. If α is a wff, so is ~α. 3. If α and β are wffs, (α.β), (α∨β), (α⊃β), and (α≡β) are wffs. (In these rules α and β are variables representing arbitrary formulas of PC.)" Implicit in the above, of course, is the requirement that if a wff contains two or more instances of any given variable — say α — then in each of those instances, α must represent the *same* concept: for otherwise the resulting formula becomes ambiguous, and can thereupon no longer be thought of as well-formed. Thus if one of the concepts in a wff is in the object-language and the other in the metalanguage, the single term α cannot represent them both.

would not be true. For instance, if **y** belongs to the metalanguage while **x** and **z** belong to the object-language, the three-term relation **x = [y; z]** would *not* be definable in **PM**.

(vi) Now Gödel continues:

> We now define a class **K** of natural numbers, as follows:
>
> **n ∈ K ≡ ~(Bew [R(n); n])** (1)
>
> (where **Bew x** means: **x** is a provable formula).

This definition should be considered carefully, for as we shall see, it is syncretic, combining object-language expressions with metalanguage expressions; and thus may not be used for establishing a sound metamathematical proof.

The meaning of this metamathematical expression[19] is,

> The natural number **n** belongs to the class **K** if and only if it is not possible to prove the formula of the system **PM** with results when the symbol[20] for the natural number **n** is substituted for the free variable that exists in **n**-th the class-sign of the system **PM**, which would be denoted by the symbol **R(n)**.

Notice that this expression cannot be defined unless it is accepted that **[R(n); n]** can be defined as Gödel intends it to be defined: and **[R(n); n]** cannot be defined as Gödel intends it to be defined unless **n** is held to simultaneously belong to two *different* languages, *viz.,* on the one hand,

(a) **n** is regarded as a symbol denoting a class-sign of the system **PM** (*i.e.*, a metalanguage symbol),

… and on the other,

(b) **n** is regarded as a symbol denoting a natural number, (*i.e.*, an object-language symbol), which is used to substitute a variable of the type of natural numbers in a class-sign of the system **PM**.

As we saw in *(iv)* above (*q.v.*), one *correct* way to write **[R(n); n]** would be **[R($\mathscr{n}$); n]** — *i.e.*, distinguishing the metalanguage term $\mathscr{n}$ from the object-language term **n**. (The metalanguage term can be written in any way that distinguishes it from the object-language term: such as for

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[19] Although it is true that the *un*-interpreted symbols of the object-language do not carry any meaning as such, that is not true of the terms and expressions of the *meta*language, which must be *about* the object-language. As such, all metalanguage expressions *must* carry a meaning; and in order for any metalanguage argument to be logically valid, there must be only *one* meaning carried by each such symbol or term.

[20] In Gödel's original German, the term used here is *Zeichen*. This has been translated by Meltzer as 'sign', but it can also be validly translated as 'symbol', and we have done so deliberately here in order to be quite clear as to what Gödel intends to say.

instance printing it in a different type-face, as above, or in a different colour, or within quotes.) But when written thus, expression (1) would have to be written as follows:

n ∈ K ≡ ~(Bew [R(*n*); n]) (1)

As such, it would be nonsensical rather than either true or false, just as the sentence "A 'grape' is a five-letter word that tastes sweet" is nonsensical. This is because it contains terms from both the object-language *and* the metalanguage.

(vii) Gödel now continues as follows:

> Since the concepts which appear in the definiens are all definable in **PM**, so too is the concept **K** which is constituted from them, …

However, as we saw in *(vi)* above (*q.v.*), Gödel's expression (1) is syncretic, containing terms from both the object-language **P** and the metalanguage **G,** and thus is not well-formed; and the concepts which appear in the definiens are *not* definable in the system **PM** — and thus neither is the concept **K**.

(viii) Now Gödel writes:

> … i.e. there is a class-sign **S**, such that the formula **[S; n]** — interpreted as to its content — states that the natural number **n** belongs to **K**.

However, as we saw in *(iv)* above (*q.v.*), the term '**[α; n]**' is not definable in the system **PM**; and thus neither is '**[S; n]**'.

(ix) After this point, the rest of Gödel's argument deteriorates quite rapidly in logical validity, for when he says:

> **S**, being a class-sign, is identical with some determinate **R(q)**, *i.e.*
>
> **S = R(q)**
>
> holds for some determinate natural number **q**.

… since there cannot be any term [**S; n]** in the object-language **PM**, Gödel does not have any reason to equate **S** with any class-sign **R(q)**. The **q** in **R(q)** belongs to the metalanguage, while **S** belongs to the object-language. Attempting to "squeeze" both into one single language would be tantamount to trying to "squeeze" both definitions of *apples* and *lemons* into one single language: the result would be nonsensical at best and self-contradictory at worst.[21]

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[21] It is always possible to render a statement self-contradictory by uniting the object-language and metalanguage levels. For example, it can be argued as follows: *[1]* Apples and lemons are words (true); *[2]* Apples and lemons are fruits (true); *[3]* Fruits are not words (true); *[4]* Therefore apples and lemons are not words (conclusion from true premises using a correct rule of inference — namely a syllogism — and which conclusion must therefore be true).

(x) Thus when Gödel adds:

> We now show that the proposition **[R(q); q]** is undecidable in **PM**.

… he is relying, to "prove" what he intends to prove, on a premise — namely expression (1) — which is syncretic, and which thus cannot be used for the sound establishment of a proof in metamathematics. *I.e.*, Gödel's reasoning is incapable of soundly establishing a proof in metamathematics, for it attempts the unification of the metalanguage **G** with the object-language **P**.

Basically, **[R(q); q]** *cannot* be undecidable in the object-language **PM** — contrary to Gödel's claim — for the simple reason that it does not *belong* to the object-language **PM**! Part of it belongs to the object-language **P** while the rest belongs to the metalanguage **G**, and thus as a whole it cannot belong to either levels of language. This is seen very clearly when it is written correctly, distinguishing object-language terms from metalanguage terms: for example as **[R(𝓆); q]**.

We can also see this from the implications of expression (1). If in expression (1), **n** were substituted by **q**, then it would read as follows — let us call it "expression (1')":

q ∈ K ≡ ~(Bew [R(𝓆); q]) (1')

But if it be assumed, hypothetically, that there is *no* difference between the **q** in **R(q)** and the **q** following the semi-colon, then if expression (1') is held to be a valid expression, its complement, *viz.*

q ∉ K ≡ Bew [R(q); q] (1a')

… must also be a valid expression.

But expression (1a'), when interpreted as to its content, contradicts itself: because its definiens, *viz.,* **Bew [R(q); q]**, when interpreted as to its content, states that **[R(q); q]** *is* provable, and therefore what **[R(q); q]** states — *viz.,* "that the natural number **q** belongs to **K**" — must be *correct.* And in that case, **q** *does* belong to **K** … which however contradicts the definiendum of expression (1a'), *viz.,* **q ∉ K**, which states that **q** does *not* belong to **K**.

Thus we see that if it be assumed, hypothetically, that there is *no* difference between the **q** in **R(q)** and the **q** following the semi-colon in the term **[R(q); q]**, then a contradiction ensues: which proves that expression (1a') cannot be a valid expression in a consistent system of logic. And since expression (1a') is derived from expression (1), this proves that expression (1) can also not be a valid expression in a consistent system of logic.

(xi) Now Gödel adds (and this is the crux of his argument):

> For supposing the proposition **[R(q); q]** were provable, it would also be correct; but that means, as has been said, that **q** would belong to **K**, i.e. according to (1), **~(Bew [R(q); q])**

Statement *[1]* directly contradicts statement *[4]* here, which is implied by it; and therefore statement *[1]* must be self-contradictory, at least by implication.

> would hold good, in contradiction to our initial assumption. If, on the contrary, the negation of **[R(q); q]** were provable, then **~(q ∈ K)**, i.e. **Bew [R(q); q]** would hold good.

… this is hardly the case, for the above argument relies on a premise, namely expression (1), which, being syncretic, *i.e.*, combining terms from both the object-language and the metalanguage, cannot be used to soundly establish a proof in metamathematics.

But there is more than one fallacy in Gödel's above argument. It is in addition to be noted that the argument Gödel is using to prove his conclusion is of the form of a *reductio ad absurdum* — or more accurately, a *reductio ad impossibile*, which, as the *Encyclopaedia Britannica* writes, is "A form of the *reductio ad absurdum* argument, known as indirect proof or *reductio ad impossibile* … one that proves a proposition by showing that its denial conjoined with other propositions previously proved or accepted leads to a contradiction."

Now there is a severe limit to such an argument, in that it *presupposes* that the proposition to be proved does not *itself* lead to a contradiction, either directly or when conjoined with other propositions previously proved or accepted! If the proposition to be proved *and* its denial *both* lead to a contradiction, directly or indirectly, then the argument falls through: for then there is no *reduction* to an absurdity or an impossibility, but merely the *re-affirmation* of an absurdity or an impossibility which was assumed or asserted *a priori*.[22]

Gödel, as we have seen, *a priori* asserts expression (1) — and that too, without any proof whatsoever of its validity — in order to *indirectly* "prove" his Theorem. But as we saw, expression (1) is syncretic, combining terms from both the object-language **P** and the metalanguage **G**, and is thus nonsensical: *i.e.*, neither true nor false.

Indeed, for precisely this reason, if Gödel's *reductio ad impossibile* were pushed further, it would run something like this:

> And supposing that **[R(q); q]** were *not* provable: then it could be written as **~(Bew [R(q); q])**, and thus, by expression (1), **q** would belong to **K** — which, of course, would *prove* that **[R(q); q]** must be correct. This contradicts our supposition that **[R(q); q]** is *not* provable; and thus **[R(q); q]** cannot be *not* provable either.

In other words, using Gödel's premises, it can be shown, using two *reductio ad impossibile* arguments, that **[R(q); q]** is *neither* provable *nor* unprovable.

---

[22] The system **PM** avoids this problem by *defining*, right from the start, that all letters used to denote elementary propositions — such as ***p***, ***q***, ***r*** and ***s*** — will denote only *true* propositions, not false ones; for in the system **PM** the '~' sign is *defined* as being identical to ***p|p***, where the stroke ('|') separates two propositions of which one or both are false. Thus '**~*p***', '**~*q***', '**~*r***' and '**~*s***' always denotes *false* propositions, never *true* ones. (See *Principia Mathematica*, Second Ed., Vol. I, page *xvi*, as also Chapter 2.) As a result, no statement that is *neither* true *nor* false is fit to be called a "proposition" of the system **PM**. However, this restriction does not apply to other systems; indeed it does not even apply to Gödel's paper, where the symbol '~' merely means "not" rather than "false" — for as we noted earlier, the sentence "Santa Claus is not asleep" is just as nonsensical (*i.e.*, neither true nor false) as "Santa Claus *is* asleep".

(xii) And thus Gödel's final words:

> **[R(q); q]** would thus be provable at the same time as its negation, which again is impossible.

… just do not apply. The formula **[R(q); q]**, being syncretic, is on the contrary neither provable nor unprovable, but rather nonsensical: *i.e.*, not a propositional formula at all.

Indeed what *would* apply would be a statement such as the following:

> **[R(q); q]** is thus *un*provable and also *not* unprovable … and this is only possible if **[R(q); q]** is itself not meaningful — *i.e.,* if it is not a "well-formed formula" — in classical symbolic logic.[23]

(xiii) In any case — and to clinch our argument — in a *consistent* system of bivalent logic such as the system **PM**, either a determinate natural number **q** *belongs* to a class **K** of natural numbers, or it *doesn't.* In such a system of logic there can be absolutely *no* other possibility; and from this it follows that at the very least, Gödel's penultimate conclusion, *viz.*, that the negation of **[R(q); q]** is not provable, cannot belong to such a system.

For if **q** *does* belong to **K**, then by expression (1), that would be equivalent to saying that **[R(q); q]** is *not* provable; and on the other hand, if **q** does *not* belong to **K**, then by the complement of expression (1), *viz.,* (1a), that would be equivalent to saying that **[R(q); q]** *is* provable.

And in a *consistent* system of bivalent logic, *there can be absolutely no other possibilities.*

To express it symbolically, from the consistency of the system **PM** it follows that

$$\mathbf{q} \in \mathbf{K} \vee \mathbf{q} \notin \mathbf{K}$$

… must hold; and there can be absolutely *no* other possibilities.

And from expression (1) it follows further that

$$\sim(\mathbf{Bew}\ [\mathbf{R(q); q}]) \vee \mathbf{Bew}\ [\mathbf{R(q); q}])$$

… must also hold. And again, there can be *no* other possibilities: either **Bew [R(q); q])** holds *or* **~(Bew [R(q); q])** holds, *but nothing else*.

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[23] It has been argued that Gödel employs Intuitionist's logic for his proof, rather than classical symbolic logic, since in Intuitionistic logic, which does not assume the excluded middle, it *is* permissible to assert **[~*p* & ~(~*p*)]** — the double negation does not collapse. However, one of the *raisons d'être* of Intuitionistic logic is that in real life, many instances of expressions of the form of the excluded middle law, **(*p* ∨ ~*p*)**, just don't hold up — neither ***p*** nor **~*p*** are true. This problem can be avoided, however, by stipulating in advance that to fit the definition of "proposition", a statement *must* be either true or false, but not neither or both. That would take care of all such statements as "Santa Claus is sleeping peacefully" (which is neither true nor false, for the simple reason that Santa Claus has no existence to begin with), and "New York is a city" (which is both true *and* false, because New York is also a State, and this ambivalence would allow one to "prove" with the help of symbols that New York is both a city and *not* a city, as follows: let **c** = "New York is a city" (which is true); **s** = "New York is a State" (which is also true); **n** = "A State is not a city" (which too is true); and from which follows "logically" that **s & n → ~c** — *i.e.*, New York is not a city — which must also be true, since it is derived from true premises using a syllogism, *viz*., a correct rule of inference.)

As a result, however, the possibility that the *negation* of **[R(q); q]** is not provable — or, in symbolic form, that **~{Bew (~[R(q); q])}** should hold — just does not *exist* in a consistent system of bivalent logic such as the system **PM**.

For note that **~{Bew (~[R(q); q])}** cannot be held equivalent to **~(Bew [R(q); q])** — or in other words, the equivalence **~{Bew (~[R(q); q])} ≡ ~(Bew [R(q); q])** cannot hold — for then its complement, which by expression **∗4.11** on page 117 of Volume I of *Principia Mathematica* (*q.v.*) must also hold, would be **Bew (~[R(q); q]) ≡ Bew [R(q); q]**: or, in Gödel's words, "**[R(q); q]** would … be provable at the same time as its negation, which … is impossible."

However, neither can **~{Bew (~[R(q); q])}** be held equivalent to **Bew [R(q); q]**, for if it did, **Bew [R(q); q]** would imply *another* contradiction, namely that **q** *does* belong to **K** — by the very meaning of **[R(q); q]** — and simultaneously, by expression (1), that **q** does *not* belong to **K**.

But the above two paragraphs exhaust the *only* two possibilities allowed by

~(Bew [R(q); q]) ∨ Bew [R(q); q]

… and thus the formula **~{Bew (~[R(q); q])}** *cannot exist at all* in a classical bivalent logic system.

And thus also, if it be hypothetically accepted, purely for the sake of argument, that Gödel *has* proved by means of a *reductio ad absurdum* that the negation of **[R(q); q]** is not provable — or, in symbolic form, that **~{Bew (~[R(q); q])}** *must* hold — then what he must have proved is, that an expression which does not even *exist* in a consistent system of bivalent classical symbolic logic *must* hold in that self-same system of logic: which is itself an absurd conclusion.

However, it is to be noted that this last conclusion, although absurd, is not altogether *impossible.* It *is* possible if the proposition **Bew ~[R(q); q]** — from which Gödel derives his proposition **~{Bew (~[R(q); q])}** by means of a *reductio ad absurdum* — is *itself* absurd.

In other words, this is only possible if Gödel *starts off* his argument with one or more absurd propositions. In which case there is no actual *reduction to* an absurdity, but merely the *reaffirmation of* one or more absurdities assumed or asserted *a priori*.[24]

[24] It should be clear to the observant reader — one familiar with the limitations of the *reductio ad absurdum* and *reductio ad impossibile* arguments — that what Gödel has *failed* to do in his 1931 Paper is to examine *all* the possibilities flowing from his *reductio ad impossibile*. In particular, he has failed to examine the possibility that **[R(q); q]** is undecidable in the system **PM** simply because it does not *belong* to the system **PM** to begin with!

PART 1-C

DETAILED DEMONSTRATION OF THE EXISTENCE IN GÖDEL'S 1931 PAPER OF THE CRITICAL ERROR OF SYNCRETISM

It should be clear to all logicians that for a metamathematical proof to be valid, each of the terms used in it must be *(a)* defined unambiguously, and *(b)* for each expression of the proof, it should be unambiguously specified as to which language — object-language or metalanguage — each of terms in the expression is defined in. These requirements, although not *exhaustive*, are without question *necessary*, for unless they are fulfilled, it would become possible to "prove" a false conclusion, or even derive a contradictory or absurd conclusion, from true premises. Ambiguity in the definition of terms — when the definitions are such as must be irreconcilably at variance with each other — must therefore absolutely invalidate any "proof" obtained therewith.[25]

As we saw above, Gödel has not defined the term **n** in his metalanguage as a natural number. One way of getting round it might be imagined to be, to define **n** in the metalanguage exactly as **n** is defined in the object-language, and then add to it a second definition, valid only in the metalanguage but not in the object-language. This, however, is not possible in the case of Gödel's Theorem, for then he would contradict himself: as we shall see hereunder.

(1) PROOF THAT GÖDEL HAS DEFINED THE TERM n IN TWO DIFFERENT WAYS IN ORDER TO ATTEMPT TO "PROVE" HIS THEOREM

We have shown above that Gödel has not distinguished between metalanguage terms and object-language terms; and we shall now show that to get around the problem created by this omission, Gödel has attempted to define the term **n** *simultaneously* in his metalanguage **G**:

(a) as a natural number exactly as it is defined in the object-language **PM**, <u>*and*</u>

(b) as a designator or indicator or representative of a class-sign of the object-language **PM**.

Indeed we shall also show that *without* attempting to do so, Gödel *cannot* attempt to prove his Theorem.

For supposing that the term **n** were defined *only* as a designator or indicator or representative of a class sign of the object-language **PM**, but *not* as a natural number (*i.e.,* not defined exactly in the way natural number is defined in the object-language **PM**), then his metalanguage expres-

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[25] Note again that in *meta*mathematics, which is *about* a formal system, terms *must* have meanings — and thus also, definitions.

sion **[R(n); n]** would become meaningless, for when "interpreted[26] as to its content" (as Gödel expresses it), it would state — to paraphrase Gödel's own words:

> that formula which is derived on replacing the free variable in the class-sign **R(n)** by the sign for the class-sign **n**.

Obviously, if **n** is *not* defined as a natural number, "the sign for the class-sign **n**" *cannot* meaningfully replace a free variable of what Gödel calls "the type of the natural numbers (class of classes)" — for the simple reason that there is no such thing as "the sign for the class-sign **n**".

On the other hand, supposing the term **n** were defined *only* as a natural number as defined in the object-language **PM**, but would *not* denote (or indicate or represent) a class-sign of the system **PM**, then Gödel's own following words, *viz.:*

> We think of the class-signs as being somehow arranged in a series, and denote the **n**-th one by **R(n)** …

… would be directly contradicted.[27]

(2) PROOF THAT SIMULTANEOUSLY DEFINING THE TERM n IN TWO DIFFERENT WAYS RENDERS GÖDEL'S "UNDECIDABLE" FORMULA [R(q); q] "NOT MEANINGFUL"

We shall now show that having been constrained to attempt to define the term **n** *simultaneously*:

(a) as a natural number exactly as that term is defined in the object-language **PM**, *and*

(b) as a designator or indicator or representative of a class-sign of the object-language **PM**,

… it can be proved that **[R(q); q]** cannot be what Gödel would have called "a meaningful formula" — or what we would nowadays call "a well-formed formula" — of a consistent system of logic like the system **PM**.

It is clear that if, with respect to any given consistent system of bivalent classical logic **L**, a formula **f** is *not* well-formed — *i.e.*, if **f** does *not* belong to **L** — then *all four* of the following would hold:

1. **f** cannot be **L-**provable,

2. **f** cannot be *not* **L**-provable,

---

[26] It is to be noted once again that we are speaking here of a *meta*language expression, and as such, it *cannot* be essentially meaningless.

[27] As mentioned earlier, although Gödel does not expressly define **n** in his paper, the term '**n**-th' can only mean "the ordinal number corresponding to the cardinal number **n**"; and the term "cardinal number" and "natural number" as used by Gödel are synonymous, since they are both a "class of classes".

3. **~f** cannot be **L**-provable, and

4. **~f** cannot be *not* **L**-provable.[28]

In Gödel's terminology this can be written as:

1. $\mathbf{Bew_L}$ **(f)** cannot hold

2. **~[**$\mathbf{Bew_L}$ **(f)]** cannot hold

3. $\mathbf{Bew_L}$ **(~f)** cannot hold, and

4. **~[**$\mathbf{Bew_L}$ **(~f)]** cannot hold.

We shall now demonstrate that this holds if **f ≡ [R(q); q]**.

Note that Gödel has already written:

> For supposing the proposition **[R(q); q]** were provable, it would also be correct; but that means, as has been said, that **q** would belong to **K**, i.e. according to (1), **~(Bew [R(q); q])** would hold good, in contradiction to our initial assumption. If, on the contrary, the negation of **[R(q); q]** were provable, then **~(q ∈ K)**, i.e. **Bew [R(q); q]** would hold good.

… or in other words, Gödel conclusively proves that **[R(q); q]** cannot be provable in a *consistent* system of logic **L**, and neither can ~**[R(q); q]** be provable in such a system.

But then again, supposing that **[R(q); q]** is *not* provable in a consistent system of logic **L**: then it could be written as **~(**$\mathbf{Bew_L}$ **[R(q); q])**, and thus it would follow, by Gödel's expression (1), that **q** belongs to **K** — which, of course, *constitutes a proof* in **L** of **[R(q); q]**, since, in Gödel's words, **[R(q); q]** "states that the natural number **n** belongs to **K**". And the very existence of such a proof contradicts the assumption that **[R(q); q]** is *not* provable.

Expressing it symbolically:

(I)	Assume **~(**$\mathbf{Bew_L}$ **[R(q); q])**	*assumption*
(II)	**~(**$\mathbf{Bew_L}$ **[R(q); q]) ≡ q ∈ K**	*by Gödel's expression* (1)
(III)	**[R(q); q] → q ∈ K**	*by Gödel's definition of* **[R(q); q]**
(IV)	**[R(q); q] → q ∈ K →** $\mathbf{Bew_L}$ **[R(q); q]**	*by Gödel's definition of* **Bew**
(V)	**~(**$\mathbf{Bew_L}$ **[R(q); q]) &** $\mathbf{Bew_L}$ **[R(q); q]**	*contradiction – by lines (I) and (IV)*

[28] Buddhist readers and scholars will surely note that this argument bears a striking resemblance to the classic tetra-lemmas used by the brilliant Indian Buddhist logician Nâgârjuna in his *Mula-Mâdhyamika-Kârikâ* or *MMK* to prove that in reality the self (in Sanskrit: *âtman*) cannot exist.

Whence it follows that the assumption in line *(I)*, namely **~(Bew$_L$ [R(q); q])**, must be false: or in other words, that **[R(q); q]** cannot be *not* provable in a consistent system of logic **L**.

And supposing that the negation of **[R(q); q]** is also not provable in a consistent system of logic **L**: then it could be written as:

~{Bew$_L$ ~[R(q); q])

… but this expression, as we already proved in *(xiii)* above (*q.v.*), cannot even *belong* to — let alone *hold* in — a consistent bivalent system of logic **L** (like for example the system **PM**).

As a consequence, *all four* of the following propositions would have to be correct:

(I) **[R(q); q]** cannot be **L**-provable — which is to say, **Bew$_L$ [R(q); q])** cannot hold
As proved by Gödel in (xi) earlier, which argument is repeated above also

(II) ~**[R(q); q]** cannot be **L**-provable — which is to say, **Bew$_L$** ~**[R(q); q])** cannot hold
As proved by Gödel in (xi) earlier, which argument is repeated above also

(III) **[R(q); q]** cannot be <u>*not*</u> **L**-provable — which is to say, ~**Bew$_L$ [R(q); q])** cannot hold
As proved by us here above

(IV) ~**[R(q); q])** cannot be <u>*not*</u> **L**-provable — which is to say, **~Bew$_L$** ~**[R(q); q])** cannot hold
As proved by us in (xiii) earlier

Thus in a *consistent* system of logic like the system **PM**, *none* of the four propositions **Bew$_L$ [R(q); q]), Bew$_L$** ~**[R(q); q])**, ~**Bew$_L$ [R(q); q])** or ~**Bew$_L$ ~[R(q); q])** can possibly hold.[29]

This, of course, is only possible if **[R(q); q]** is *not* what Gödel in his days might have called a "meaningful formula" — or what we in our days would call a "well-formed formula" — of a consistent system of logic like the system **PM**.

And since **[R(q); q]** is but a particular instance of **[R(n); n]**, the expression **[R(n); n]** must also not be a well-formed formula of a consistent system of logic like the system **PM** …

… *Q.E.D.*

[29] Note also that the law of double negation — *viz.*, ***p* ≡ ~(~*p*)** — only holds in *consistent* system of bivalent logic: and there too, only for formulae that actually *belong* to that system. It does *not* hold for an inconsistent system of logic, and neither does it hold for formulae that are not themselves part of a consistent system of bivalent logic.

PART 2-A

CRITIQUE OF *SECTION 2* OF KURT GÖDEL'S 1931 PAPER

We shall now critique ***Section 2*** of Gödel's above-mentioned 1931 paper: that is to say, the so-called "rigorous development of the proof sketched above". This we shall do, in general, in the same fashion as we have critiqued ***Section 1*** of it: that is to say, again by analysing his expressions and words term by term and sentence by sentence, so as to lay bare the assumptions concealed in the purported proof, and thereby reveal its logical fallacies (which, as we shall show, are essentially the same as those of what Gödel calls the "main lines of the proof").

However, where there is no pressing need to go over a passage sentence by sentence — as for instance at the beginning, where Gödel merely sets forth definitions, axioms and so on — we shall simply quote the relevant passage and give our general comments.

In this part of our critique, namely **PART 2-A**, we shall deal with the preliminary portion of Gödel's ***Section 2***, wherein he merely sets the groundwork for what he refers to as "the rigorous development of the proof sketched above".

GÖDEL'S DEFINITIONS OF THE "BASIC SIGNS OF THE SYSTEM P"

(xiv) Gödel's words in ***Section 2*** of his paper begin as follows:

> We proceed now to the rigorous development of the proof sketched above, and begin by giving an exact description of the formal system **P**, for which we seek to demonstrate the existence of undecidable propositions. **P** is essentially the system obtained by superimposing on the Peano axioms the logic of **PM** (numbers as individuals, relation of successor as undefined basic concept).
>
> The basic signs of the system **P** are the following:
>
> I. Constants: "**~**" (not), "∨" (or), "∀" (for all),[30] "**0**" (nought), "***f***" (the successor of), "**(**", "**)**" (brackets).
>
> II. Variables of first type (for individuals, i.e. natural numbers including 0): "$\mathbf{x_1}$", "$\mathbf{y_1}$", "$\mathbf{z_1}$", …
>
> III. Variables of second type (for classes of individuals): "$\mathbf{x_2}$", "$\mathbf{y_2}$", "$\mathbf{z_2}$", …

[30] Gödel in his original German paper uses the symbol 'Π' for this, but we shall use the more common modern symbol '∀'.

IV. Variables of third type (for classes of classes of individuals): "$\mathbf{x_3}$", "$\mathbf{y_3}$", "$\mathbf{z_3}$", …

and so on for every natural number as type.

Note: Variables for two-termed and many-termed functions (relations) are superfluous as basic signs, since relations can be defined as classes of ordered pairs and ordered pairs again as classes of classes, e.g. the ordered pair **a**,**b** by **((a)**,**(a,b))**, where **(x**,**y)** means the class whose only elements are **x** and **y**, and **(x)** the class whose only element is **x**.

Up to this point Gödel has merely given us an outline of the basic signs *of* the object-language, the system **P**. It is to be borne in mind that according to the use / mention rules of establishing a valid proof in a metalanguage, it would not be permissible to *use* the above in the metalanguage **G**, but only to *mention* them.

It may also be noted that Gödel is not too particular as to the use of quote marks around the terms of the object-language **P** which he mentions here above. At times, later in his Paper, he uses double quote marks around them, and at other times he omits the double quote marks. (At no place does he use single quote marks.)

SOME OF GÖDEL'S OWN DEFINITIONS

(xv) Gödel now defines some additional terms as follows:

By a **sign of first type** we understand a combination of signs of the form:

a, ***f*a**, ***ff*a**, ***fff*a** … etc.

where **a** is either **0** or a variable of first type. In the former case we call such a sign a **number-sign**. For **n > 1** we understand by a **sign of n-th type** the same as **variable of n-th type**. Combinations of signs of the form **a(b)**, where **b** is a sign of **n**-th and **a** a sign of **(n+1)**-th type, we call **elementary formulae**. The class of **formulae** we define as the smallest class containing all elementary formulae and, also, along with any **a** and **b** the following: **~(a)**, **(a)**∨**(b)**, **x∀(a)** (where **x** is any given variable). We term **(a)**∨**(b)** the **disjunction** of **a** and **b**, **~(a)** the **negation** and **x∀(a)** a **generalization** of **a.** A formula in which there is no free variable is called a **propositional formula** (**free variable** being defined in the usual way). A formula with just **n** free individual variables (and otherwise no free variables) we call an **n-place relation-sign** and for **n = 1** also a **class-sign**.

By **Subst a(v|b)**[31] (where **a** stands for a formula, **v** a variable and **b** a sign of the same type as **v**) we understand the formula derived from **a**, when we replace **v** in it, wherever it is free, by **b**. We say that a formula **a** is a type-lift of another one **b**, if **a** derives from **b**, when we increase by the same amount the type of all variables appearing in **b**.

[31] This kind of formula is given in Gödel's original Paper as **Subst a**$\left(\begin{smallmatrix}v\\c\end{smallmatrix}\right)$. However, since this is typographically very laborious, we shall use Meltzer's on-line convention and express such formulae as **Subst a(v|c)**.

Here Gödel is mostly defining, in bold type, the terms of the *metalanguage* ***G***. However, he is obviously not too particular as to specifying which terms in bold type belong to which language: object-language or metalanguage. For example, he mentions the term '**n**' which denotes a 'natural number' in the object-language **P** .

As we shall see in **PART 2-B** of our critique, this results in a unification of the object-language level with the metalanguage level. This is not permissible for the establishment of a sound metamathematical theorem. If it is done, in fact, it ends up by Gödel enunciating a proposition, *viz.,* his expression (8.1), which is self-contradictory, and also contradicts another of Gödel's own propositions, *viz.,* his **Proposition V**, in order to establish his Theorem.

GÖDEL'S DEFINITIONS OF "AXIOMS"

(xvi) The same comment applies to what follows of Gödel's paper hereunder:

> The following formulae (I-V) are called **axioms** (they are set out with the help of the customarily defined abbreviations: ., ⊃, ≡, **(∃x)**, =, and subject to the usual conventions about omission of brackets):
>
> I. **1.** $\sim(f x_1 = 0)$
> **2.** $f x_1 = f y_1 \supset x_1 = y_1$
> **3.** $x_2(0).x_1 \forall (x_2(x_1) \supset x_2(f x_1)) \supset x_1 \forall (x_2(x_1))$
>
> II. Every formula derived from the following schemata by substitution of any formulae for **p**, **q** and **r**.
>
> **1.** $p \vee p \supset p$
> **2.** $p \supset p \vee q$
> **3.** $p \vee q \supset q \vee p$
> **4.** $(p \supset q) \supset (r \vee p \supset r \vee q)$
>
> III. Every formula derived from the two schemata
>
> **1.** $v \forall (a) \supset$ **Subst** $a(v|c)$[32] **2.** $v \forall (b \supset a) \vee b \supset v \forall (a)$
>
> by making the following substitutions for **a**, **v**, **b**, **c** (and carrying out in 1 the operation denoted by "**Subst**"): for **a** any given formula, for **v** any variable, for **b** any formula in which **v** does not appear free, for **c** a sign of the same type as **v**, provided that **c** contains no variable which is bound in **a** at a place where **v** is free.
>
> IV. Every formula derived from the schema

---

[32] ***N.B.*** In Meltzer's on-line translation this axiom is given as '**v ∀ (a)** ∨ **Subst a(v|c)**', but this is a typographical error. However, in his printed copy it is correctly given.

1. (∃u)(v ∀ (u(v) ≡ a))

on substituting for **v** or **u** any variables of types **n** or **n + 1** respectively, and for **a** a formula which does not contain **u** free. This axiom represents the axiom of reducibility (the axiom of comprehension of set theory).

V. Every formula derived from the following by type-lift (and this formula itself):

1. $\mathbf{x_1 \forall (x_2(x_1) \equiv y_2(x_1)) \vee x_2 = y_2}$.

This axiom states that a class is completely determined by its elements.

A formula **c** is called an **immediate consequence** of **a** and **b**, if **a** is the formula **(~(b))** ∨ **(c)**, and an **immediate consequence** of **a**, if **c** is the formula **v ∀ (a)**, where **v** denotes any given variable. The class of **provable formulae** is defined as the smallest class of formulae which contains the axioms and is closed with respect to the relation "immediate consequence of".

It will again be noted that Gödel does not differentiate above between definitions of the object-language (*i.e.*, the system **P**) and those of his metalanguage **G**.

GÖDEL'S "ASSIGNING" OF NUMBERS TO BASIC SIGNS AND SERIES OF SIGNS OF THE SYSTEM P

(xvii) Now Gödel comes to the point in his paper which ultimately brings about the syncretism we spoke about in **PART 1** of our critique: namely, the assigning of numbers in one-to-one correspondence to symbols, formulae and proofs of the object-language **P** in order to identify or designate them. As Gödel writes:

The basic signs of the system **P** are now ordered in one-to-one correspondence with natural numbers, as follows:

"**0**" … 1
"*f*" … 3
"**~**" … 5
"∨" … 7
"∀" … 9
"**(**" … 11
"**)**" … 13

Furthermore, variables of type **n** are given numbers of the form $\mathbf{p^n}$ (where **p** is a prime number > 13). Hence, to every finite series of basic signs (and so also to every formula) there corresponds, one-to-one, a finite series of natural numbers.

This is how the so-called "Gödel-numbers" begin to be defined — and here is where the error of syncretism begins to creep into Gödel's purported "proof".

As we saw earlier in **PART 1** of our critique, the last sentence of Gödel's quoted above is just not correct: this finite series of symbols — namely '**1**', '**3**', '**5**', '**7**', '**9**', '**11**' and '**13**' — are here defined *in the metalanguage **G**;* and in that language they are defined, *not* as natural numbers (as defined in the system **P**), but rather as *designators* or *indicators* or *labels* or *names* of the basic signs of the system **P** (*i.e.,* of the object-language).

It would perhaps be best to be quite clear about this; and so we add hereunder a few more lines of clarification.

Symbols such as '**1**', '**2**', '**3**', '**4**', '**5**', '**13**', '**17**', '**23,389**', *etc.* are *by themselves* merely symbols, and do not represent anything. They only come to represent some concept when they are *defined* as representing that particular concept, by an explicit or implicit process of *interpreting* the given symbols.

In the *formal* system **P** also — if **P** is taken as a *purely* formal system — these symbols do not represent anything. But in *mathematics* — *i.e.*, when the formal system **P** is *interpreted* so as to formalise the proofs of mathematics — they come to represent the concepts which we call "natural numbers" (also called "counting numbers", since they are used for *counting* things).

Natural numbers are *concepts: i.e.*, mental objects. They can be neither seen, heard, touched, tasted or smelled. The symbols '**1**', '**2**', '**3**', '**4**', '**5**', '**13**', '**17**', '**23,389**', *etc.*, on the other hand, are visible *material* objects: they appear as symbols on paper, on computer screens, or other such material objects. As such, they are not *themselves* natural numbers. They merely *represent* natural numbers — and that too, only when so defined.

When otherwise defined, symbols such as '**1**', '**2**', '**3**', '**4**', '**5**', '**13**', '**17**', '**23,389**', *etc.* can also be used to represent some concept other that natural numbers. In the TV series *Star Trek: Voyager* there is a character by the name of '**7**', sometimes also called '**7**-of-**9**'. Here, '**7**' does not represent a number, natural or otherwise: it represents the *name* of this particular character.

In ordinary life such symbols are also used to represent things other than numbers: they may represent such diverse things as telephone country codes or area codes, US zip codes, credit-card "numbers" and Israeli automobile licence plate "numbers".[33] In such cases, they do *not* represent natural numbers as defined in the system **P** (*i.e.*, when the system **P** is interpreted so as to derive mathematical proofs from it.) They represent a particular geographical area, credit card, or a particular Israeli automobile — as the case may be — which is what they designate or identify; but they are not capable of designating or identifying any *quantity* at all — they cannot be used for counting things. They do not, that is, belong to number theory.

We realise this from the fact that telephone "numbers" or US zip codes are not numbers in the mathematical sense, capable of being meaningfully added together, subtracted from one another, having their square roots or cube roots extracted, *etc., etc.:* when mathematical operations such as these are performed on telephone "numbers", the result is utterly nonsensical.

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[33] In Israel, automobile licence plate numbers are not represented alpha-numerically (because of the three major differing alphabets in use in that part of the world), but purely numerically.

We also realise this when we remind ourselves that such things as telephone numbers, credit card numbers and license plate numbers can be expressed alpha-numerically, or even entirely without any digits. They can be expressed entirely alphabetically, or even in Chinese characters or Egyptian hieroglyphics. They would still fulfil their function as *indicators* or *labels* or *names* when so expressed.

Similarly, Gödel's metalanguage symbols '**1**', '**3**', '**5**', '**7**', '**9**', '**11**' and '**13**' above do not represent or designate natural numbers at all — which is to say, they do not designate the concept of a *quantity*, with which mathematical operations can be meaningfully carried out, and mathematical proofs obtained thereupon. They merely label or designate, in his metalanguage **G**, the symbols '**0**', '***f***', '~', '∨', '∀', '**(**' and '**)**' of the object-language **P**. And the same applies to the symbols expressed by Gödel as "$\mathbf{p}^{\mathbf{n}}$ (where **p** is a prime number > 13)".

Unfortunately, both the numbers of number theory, as well as "numbers" such as telephone numbers, licence plate numbers, page numbers, or credit-card numbers, are commonly spoken of as "numbers". This results in an ambiguity of which Gödel has taken full advantage. But if properly analysed as above, it is seen that these are two *different* definitions of the word "number".

As we showed in **PART 1**, *'apple'* can be defined as either a fruit or as a word … but not as both, on pain of leading to absurdities and even contradictions. Likewise, symbols such as '**1**', '**3**', '**5**', '**7**', '**9**', '**11**', '**13**', *etc.* can be defined *either* as natural numbers *or* as designators or numerical names of signs and series of signs of the system **P**. But not as both: for that would be tantamount to defining *'apple'* as both a fruit *and* a word … which two definitions, being incompatible, result in a syncretism leading to contradictions.

For the sake of brevity we shall in our critique occasionally refer to his above-described method of assigning symbols that merely *look* like natural numbers to basic signs — and series of basic signs — of the system **P**, as "Gödel-numbering" or "Gödelisation". We shall also show that this process of "Gödel-numbering" or "Gödelisation" is one of the principal errors Gödel makes.

That is to say, he makes the error of implying tacitly that "Gödel-numbers" are a subset of natural numbers. They are not. By the definition of "subset", every element of a subset ***N*** must be a member of the set ***M*** of which ***N*** is a subset; so that if an element **n**, say, is a member of ***N***, then **n** must also be a member of ***M***. But the "Gödel-number" **5**, say, is *not* a member of the set of natural numbers. The natural number **5** — *qua* natural number — is *not* defined as corresponding to the symbol '~' of the system **P** … as the Gödel-number **5** is.[34] The former belongs to one set of terms, *viz.,* the set constituting the object-language **P**, while the latter belongs to quite another set of terms, *viz.,* the set constituting the metalanguage **G**. And as we saw from the use / mention rules for soundly establishing a proof in a metalanguage, it is not permissible to *use* any of the former set of terms *in* the latter.

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[34] Strictly speaking the Gödel-number of the symbol '~' is not **5** but $\mathbf{2^5}$, as we shall see in *(xviii)* below. But for the purposes of the present argument it is immaterial which of the two is considered to be the Gödel-number of '~'.

So "Gödel-numbers" can never be a sub-set of natural numbers, because the set of "Gödel-numbers" is a sub-set of the statements (*i.e.*, formulae, propositions, *etc.*) of the metalanguage **G**, while the set of natural numbers is a sub-set of the statements (*i.e.*, formulae, propositions, *etc.*) of the object-language **P**. Just as the words *'apples'* and *'lemons'* can never be elements of the set of fruits, because the words *'apples'* and *'lemons'* belong to the metalanguage while the corresponding fruits belong to the object-language, so too Gödel-numbers can never be a elements of the set of natural numbers, because Gödel-numbers belong to the metalanguage, while natural numbers — along with all the other symbols and formulae of the system **P** to which Gödel-numbers correspond — belong to the object-language.[35]

This point should be borne firmly in mind, because it is the very crux of our critique, and the principal error Gödel makes.

We would see this even more clearly if we were to assign a unique natural number to every natural number by some easily-devised method. For example, if — to paraphrase Gödel — we were to write:

> Natural numbers are now ordered in one-to-one correspondence with natural numbers, as follows:
>
> "0" … **1**
> "1" … **2**
> "2" … **3**
> "3" … **4**
> "4" … **5**
> "5" … **6**
> "6" … **7**
>
> … and so on for all natural numbers, the number assigned to each natural number being one more than the natural number to which it is assigned.

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[35] An objection has been raised in this regard that a Gödel-number, being a product of primes, must be a natural number just as any product of primes is. By this argument, just as even numbers or prime numbers are subsets of the set of natural numbers, so Gödel-numbers, being products of primes, should also be considered to be a subset of the natural numbers. It is of course true that the even number **2** the same as the natural number **2**, and the prime number **11** the same as the natural number **11**; and it is true that in each of these cases there is a predicate that is a recursive function — an algorithm — that identifies what even numbers and prime numbers are, and each of these predicates identifies a different subset of the natural numbers. However, these predicates are implicit in, and directly follow from, the definition of "natural number". **2** is an even number and **11** a prime number simply by virtue of **2** and **11** being natural numbers. This is shown by the fact that **2** can never be an odd number nor can **11** ever be non-prime, even if they are so "defined": such "definitions" would contradict that which is implicit in the definition of "natural number". That is because all these definitions belong to the object-language. The definition of "Gödel-number", however, does *not* follow from the definition of "natural number", and thus the predicate "Gödel-number" is *not* implicit in the predicate "natural number": a Gödel-number is *not* one simply by virtue of being a natural number. And this is because Gödel-numbers *by definition* belong to the metalanguage. (This is similar to the way a word can also be either a noun or a pronoun or a conjunction or an adjective or a verb or an adverb. But a fruit is not a word, and so '*apple*' the word is not the same thing as *apple* the fruit.)

Here, we shall assume that the object-language numbers are on the left and the metalanguage "numbers" are on the right. Now it is obvious that the "numbers" so assigned are not equivalent to the numbers *to* which they are assigned: that is to say, it would be utterly meaningless to include both types of numbers in any single formula. If the metalanguage numbers and the object-language "numbers" were not distinguished from one another — *i.e.*, if they were written the same way — contradictions would easily arise. For example, it would be possible to "prove" that 3 + 4 = 8, if 4 and 8 belong to the object-language and 3 to the metalanguage … and also to "prove" its negation, namely 3 + 4 ≠ 8, if all of the numbers belong to the object-language. In short, any such "formula" or "equation" would be ambiguous, and therefore belong to an inconsistent system of logic: capable of proving a formula as well as its negation, *i.e.*, of deriving self-contradictory conclusions.

In this regard, we particularly draw the reader's attention to the fact that although Gödel writes:

> "The basic signs of the system **P** are now ordered in one-to-one *correspondence*[36] with natural numbers"

… in his purported proof he actually requires, not *merely* that the numerical symbols **1**, **3**, **5**, **7**, **9**, **11** and **13** *correspond* respectively to the basic signs "**0**", "***f***", "~", "∨", "∀", "**(**" and "**)**", but that they respectively be *representative* of them and even be considered the *equivalent* of them: in the sense that the ones may be substituted for the corresponding others *in* the very formulae to which they correspond. This has already become apparent in ***Section 1*** of his 1931 Paper (*q.v.*), wherein he requires the natural number **q** to be substituted for the free variable *in* the formula **R(q)**, that formula in turn being required to correspond to the natural number **q**. It will also become apparent — albeit by implication only — from *(xxii)* below *et seq.* (*q.v.*), and especially when discussing Gödel's expressions (11) to (13).

And yet it will also become apparent from Gödel's requirements in *(xxiv)* below *et seq.* (*q.v.*) that the numerical symbols **1**, **3**, **5**, **7**, **9**, **11** and **13** *cannot* respectively be representative of, or equivalent to, the basic signs "**0**", "***f***", "~", "∨", "∀", "**(**" and "**)**".

As a result, Gödel — at least by implication — will be seen to completely and absolutely contradict himself, which renders his method of proof self-contradictory, and therefore invalid in any known system of logic.[37]

(xviii) Gödel now continues:

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[36] Our emphasis.

[37] As mentioned earlier in a footnote in **PART 1** of our critique, although there are many valid kinds of logic, *there does not exist even one single kind of logic* which permits complete and absolute contradictions to be derived within it. (*Partial* contradictions are permitted in some kinds of multivalent logic — such as Fuzzy Logic — but the above contradiction is not a partial contradiction; and in any case, we are not concerned with such logics here, but only with bivalent logic.)

> These finite series of natural numbers we now map (again in one-to-one correspondence) on to natural numbers, by letting the number $\mathbf{2^{n_1} \cdot 3^{n_2} \cdot \ldots p_k^{n_k}}$ correspond to the series $\mathbf{n_1}$, $\mathbf{n_2}$, … $\mathbf{n_k}$, where $\mathbf{p_k}$ denotes the **k**-th prime number in order of magnitude. A natural number is thereby assigned in one-to-one correspondence, not only to every basic sign, but also to every finite series of such signs.

This merely compounds the original error, for as we saw in *(xvii)* above, in the first place, the *soi-disant* "finite series of natural numbers" is not a series of natural numbers at all — *i.e.*, they do not belong to the object-language **P** — but are a finite series of *numerical names* or *labels* belonging to the metalanguage **G**.

That is to say, Gödel here above merely designates numerical names using *additional* numerical names. This is merely naming a name: and the name of a name is no more the actual *object* that is named, than the name of the object is the actual object that is named.[38]

Here, for example, the finite series of basic signs '**(0)**' of the object-language **P** would be denoted (or designated) by the series of so-called "natural numbers '**11, 1, 13**' … which in turn would be "mapped" on to the so-called "natural numbers" '$\mathbf{2^{11} \cdot 3^{1} \cdot 5^{13}}$', which "product", if worked out, comes to '$\mathbf{7.5x10^{12}}$'.[39] Of course, as shown earlier, this is not a natural number belonging to the system **P** at all, but a *numerical name* or *label* of the series of symbols '**(0)**' of the system **P**. As such, this label belongs to the metalanguage **G**, not to the object-language **P**.

(xix) Gödel now says:

> We denote by **Φ(a)** the number corresponding to the basic sign or series of basic signs **a**.

As we saw above, this term, *viz.* '**Φ(a)**', obviously does not represent a number at all — in the sense of a natural number as defined when the system **P** is interpreted in mathematical terms — but rather represents a symbol (or series of symbols) in the metalanguage **G** corresponding to a series of basic signs of the object-language **P**. The term '**Φ(a)**' is simply the representative of a *code*, a *designator*, an *identifier*, a *label*, a *name* expressed in digits: since it belongs to the metalanguage **G**, it does not — and *cannot* — represent a number belonging to the system **P**.

That is to say, it is not capable of designating or identifying a particular *quantity* on which mathematical functions such as addition, subtraction, division, *etc*. may validly be performed.

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[38] Just as objects can be given names, so can names be given names. For instance, the name of the President of the United States is — as of the date of beginning our critique — 'Bill Clinton'. Now one could give a name to the *name* 'Bill Clinton': for example, the name 'Jupiter'. But in that case, the name 'Jupiter' would not denote the *person* Bill Clinton, but rather would denote the *name* 'Bill Clinton'. In other words, both 'Bill Clinton' and 'Jupiter' are names; but as defined above, the former is the name of a *person*, while the latter is the name of a *name*.

[39] However, such a multiplication cannot logically hold up, for in order to assume that it can, we have also to assume that the series of symbols '$\mathbf{2^{n_1}}$', '$\mathbf{3^{n_2}}$' … '$\mathbf{p_k^{n_k}}$' *etc.* indicate or represent natural numbers that *may* be multiplied together according to the rules of the system **P**. But since the symbols series '$\mathbf{n_1}$,', '$\mathbf{n_2}$', … '$\mathbf{n_k}$' *etc.* do not indicate or denote natural numbers at all, neither can the series of symbols '$\mathbf{2^{n_1}}$', '$\mathbf{3^{n_2}}$' … '$\mathbf{p_k^{n_k}}$' *etc.* do so; and thus they cannot validly or meaningfully be multiplied together, any more than telephone numbers or chapter numbers can be.

So when Gödel continues:

> Suppose now one is given a class or relation $\mathbf{R(a_1,a_2,\ldots a_n)}$ of basic signs or series of such. We assign to it that class (or relation) $\mathbf{R'(x_1,x_2,\ldots x_n)}$ of natural numbers, which holds for $\mathbf{x_1, x_2, \ldots x_n}$ when and only when there exist $\mathbf{a_1, a_2, \ldots a_n}$ such that $\mathbf{x_i=\Phi(a_i)}$ (**i**=1,2,…n) and $\mathbf{R(a_1,a_2,\ldots a_n)}$ holds.

… obviously there cannot exist any '$\mathbf{a_1, a_2, \ldots a_n}$' such that $\mathbf{x_i=\Phi(a_i)}$ (**i**=1,2,…n) holds, for there *can be no equivalence or identity* between $\mathbf{x_i}$ — which may be substituted by a genuine natural number as defined in the system **P**, and which therefore belongs to the object-language — and $\Phi(\mathbf{a_i})$, which is a mere designator or identifier of a series of basic signs of the system **P**, and thus belongs to the metalanguage **G**. In other words, since $\Phi(\mathbf{a_i})$ does not belong to the object-language **P**, it is not a natural number at all.

And as a consequence — namely as a result of the fact that the equivalence or identity relation $\mathbf{x_i=\Phi(a_i)}$ cannot possibly hold — there can *be* no "class (or relation) $\mathbf{R'(x_1,x_2,\ldots x_n)}$ of natural numbers … which holds for $\mathbf{x_1, x_2, \ldots x_n}$ when and only when there exist $\mathbf{a_1, a_2, \ldots a_n}$ such that $\mathbf{x_i=\Phi(a_i)}$ (**i**=1,2,…n) and $\mathbf{R(a_1,a_2,\ldots a_n)}$ holds" (as Gödel expresses it)![40]

(xx) Now Gödel says:

> We represent by the same words in SMALL CAPITALS[41] those classes and relations of natural numbers which have been assigned in this fashion to such previously defined metamathematical concepts as "variable", "formula", "propositional formula", "axiom", "provable formula", etc. The proposition that there are undecidable problems in the system **P** would therefore read, for example, as follows: There exist PROPOSITIONAL FORMULAE **a** such that neither **a** nor the NEGATION of **a** are PROVABLE FORMULAE.

Here Gödel tries to distinguish between metamathematical concepts and mathematical concepts by putting the latter in SMALL CAPITALS. Thus the term PROVABLE FORMULA means "provable formula of the system **P**".

However, this effort at separating the terms of the object-language from those of the metalanguage is utterly ineffectual, because the only terms that are separated are those expressed in natural language, and not in formal language (*i.e.*, in symbols). We shall see the consequences of this in greater detail further on.

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[40] Taking our earlier example from *(xviii)* above, if we take the series of basic signs as '**(0)**', then '**Φ(a)**' would be, according to Gödel's method described by him, the *numerical name* of the series of numerical names '**11,1,13**' — which in turn would be $\mathbf{2^{11} \cdot 3^{1} \cdot 5^{13}}$ … and the result would be '**7,500,000,000,000**' (*i.e.,* '**7.5 trillion**'.) But this '**7.5 trillion**', being defined *in the metalanguage **G***, is not a number equivalent to, say, the genuine natural number (*i.e.,* a natural number as defined in the object-language **P**) 1,000 x 1,000 x 1,000 x 1,000 x 7.5, but rather the *designator* or *identifier* of the class — as defined in the object-language **P** — whose only member is nought!

[41] In Meltzer's translation, these words are in *italics*. However, in our critique we shall revert to Gödel's original practice of using SMALL CAPITALS, and use *italics* only in the ordinary way for emphasis, as well as for non-English terms and abbreviations, and for roman numerals denoting the different points of our critique.

GÖDEL'S DEFINITION OF "RECURSIVE"

(xxi) Now Gödel continues at somewhat of a tangent, as follows:

> We now introduce a parenthetic consideration having no immediate connection with the formal system **P**, and first put forward the following definition: A number-theoretic function $\phi(x_1,x_2,\ldots x_n)$ is said to be **recursively defined** by the number-theoretic functions $\psi(x_1,x_2,\ldots x_{n-1})$ and $\mu(x_1,x_2,\ldots x_{n+1})$, if for all $x_2, \ldots x_n, k$ the following hold:
>
> $$\phi(0,x_2,\ldots x_n) = \psi(x_2,\ldots x_n)$$
> $$\phi(k+1,x_2,\ldots x_n) = \mu(k,\phi(k,x_2,\ldots x_n),x_2,\ldots x_n). \quad (2)$$
>
> A number-theoretic function ϕ is called recursive, if there exists a finite series of number-theoretic functions $\phi_1, \phi_2, \ldots \phi_n$ which ends in ϕ and has the property that every function ϕ_k of the series is either recursively defined by two of the earlier ones, or is derived from any of the earlier ones by substitution, or, finally, is a constant or the successor function $x+1$. The length of the shortest series of ϕ_i, which belongs to a recursive function ϕ, is termed its **degree**. A relation $R(x_1,x_2,\ldots x_n)$ among natural numbers is called recursive, if there exists a recursive function $\phi(x_1,x_2,\ldots x_n)$ such that for all $x_1, x_2, \ldots x_n$
>
> $$R(x_1,x_2,\ldots x_n) \equiv [\phi(x_1,x_2,\ldots x_n) = 0].$$

Note that the functions $\phi(x_1,x_2,\ldots x_n)$, $\psi(x_1,x_2,\ldots x_{n-1})$ and $\mu(x_1,x_2,\ldots x_{n+1})$ are, according to Gödel, all "number-theoretic" functions. Number theory, of course, is defined and enunciated in the object-language, *viz.,* the system **P**: indeed it is one of the main purposes of the system **P** to define and enunciate number theory. Thus if the above functions are to be defined as "number-theoretic functions" *in the metalanguage **G*** as well, number theory must *also* be defined in the metalanguage **G**; and there it must be defined *exactly as it is defined in the object-language **P***.

If so, then in the metalanguage **G** the symbols 'x_1', 'x_2', … 'x_n' must all represent natural numbers (or classes of natural numbers) *as natural numbers are defined in number theory* — and that too, *as 'number theory' is defined in the mathematical interpretation of the object-language **P*** — and *not* as designators or identifiers of class-signs *of* the object-language **P**.[42] But obviously that is not permissible, since the above-noted functions $\phi(x_1,x_2,\ldots x_n)$, $\psi(x_1,x_2,\ldots x_{n-1})$ and $\mu(x_1,x_2,\ldots x_{n+1})$ belong to the object-language, and thus according to the use / mention rules for soundly establishing a metamathematical proof, they may *not* be used in the metalanguage **G**.

(xxii) Gödel now continues as follows:

[42] This is according to the principle of "substituting equals for equals". Without such a principle, of course, one could "equate" anything with anything! However, as we shall see (and as we have also seen earlier), Gödel intends the symbols 'x_1', 'x_2', … 'x_n' to be simultaneously defined *both* as natural numbers are defined in number theory, *and* as designators or identifiers of class-signs of the object-language **P**. This is an ambiguity, and ends up resulting in the fallacy of syncretism.

The following propositions[43] hold:

I. *Every function (or relation) derived from recursive functions (or relations) by the substitution of recursive functions in place of variables is recursive; so also is every function derived from recursive functions by recursive definition according to schema (2).*

II. *If* **R** *and* **S** *are recursive relations, then so also are* **~R**, **R** ∨ **S** *(and therefore also* ***R & S****).*

III. *If the functions* $\phi(\chi)$ *and* $\psi(\eta)$ *are recursive, so also is the relation:* $\phi(\chi) = \psi(\eta)$.

IV. *If the function* $\phi(\chi)$ *and the relation* $\mathbf{R}(\mathbf{x},\eta)$ *are recursive, so also then are the relations* **S**, **T**

$\mathbf{S}(\chi,\eta) \sim (\exists \mathbf{x})[\mathbf{x} \leq \phi(\chi)\ \&\ \mathbf{R}(\mathbf{x},\eta)]$
$\mathbf{T}(\chi,\eta) \sim (\mathbf{x})[\mathbf{x} \leq \phi(\mathbf{c}) \supset \mathbf{R}(\mathbf{x},\eta)]$

and likewise the function ψ

$\psi(\chi,\eta) = \varepsilon \mathbf{x}\ [\mathbf{x} \leq \phi(\mathbf{c})\ \&\ \mathbf{R}(\xi,\eta)]$

where $\varepsilon \mathbf{x}\ \mathbf{F}(\mathbf{x})$ means: the smallest number **x** for which **F(x)** holds and 0 if there is no such number.

Proposition I follows immediately from the definition of "recursive". Propositions II and III are based on the readily ascertainable fact that the number-theoretic functions corresponding to the logical concepts **~**, ∨, **=**

$$\alpha(\mathbf{x}),\ \beta(\mathbf{x},\mathbf{y}),\ \gamma(\mathbf{x},\mathbf{y})$$

namely

$\alpha(0) = 1;\ \alpha(x) = 0$ **for** $x \neq 0$
$\beta(0,x) = \beta(x,0) = 0;\ \beta(x,y) = 1,$ **if** x, y **both** $\neq 0$
$\gamma(x,y) = 0,$ **if** $x = y;\ \gamma(x,y) = 1,$ **if** $x \neq y$

are recursive. The proof of Proposition IV is briefly as follows: According to the assumption there exists a recursive $\rho(\mathbf{x},\eta)$ such that

$$\mathbf{R}(\mathbf{x},\mathbf{h}) \equiv [\rho(\mathbf{x},\mathbf{h}) = 0].$$

[43] The German word Gödel uses here — which has been translated by Meltzer as "propositions" — is *Sätze*, which can also be validly translated as "theorems". In many translations of Gödel's Paper, in fact, **Propositions I-IV** are called **Theorems I-IV**. Both translations are linguistically correct. We however shall retain Meltzer's term "propositions", because a Theorem is only rightly so called if it is *proven* — whereas in our critique we shall demonstrate that Gödel's **Proposition VI**, which constitutes the crux of his 1931 Paper, is *not* proven.

We now define, according to the recursion schema (2), a function $\mathrm{X}(\mathbf{x},\eta)$ in the following manner:

$$\mathrm{X}(\mathbf{0},\eta) = \mathbf{0}$$
$$\mathrm{X}(\mathbf{n+1},\eta) = (\mathbf{n+1}).\mathbf{a} + \mathrm{X}(\mathbf{n},\eta).\alpha(\mathbf{a})$$

where

$$\mathbf{a} = \alpha[\alpha(\rho(\mathbf{0},\eta))].\alpha[\rho(\mathbf{n+1},\eta)].\alpha[\mathrm{X}(\mathbf{n},\eta)].$$

$\mathrm{X}(\mathbf{n+1},\eta)$ is therefore either $= \mathbf{n+1}$ (if $\mathbf{a} = 1$) or $= \mathrm{X}(\mathbf{n},\eta)$ (if $\mathbf{a} = 0$). The first case clearly arises if and only if all the constituent factors of **a** are 1, i.e. if

$$\sim\mathbf{R}(\mathbf{0},\eta)\ \&\ \mathbf{R}(\mathbf{n+1},\eta)\ \&\ [\mathrm{X}(\mathbf{n},\eta) = \mathbf{0}].$$

From this it follows that the function $\mathrm{X}(\mathbf{n},\eta)$ (considered as a function of **n**) remains 0 up to the smallest value of **n** for which $\mathbf{R}(\mathbf{n},\eta)$ holds, and from then on is equal to this value (if $\mathbf{R}(\mathbf{0},\eta)$ is already the case, the corresponding $\mathrm{X}(\mathbf{x},\eta)$ is constant and = 0). Therefore:

$$\psi(\chi,\eta) = \mathbf{C}(\phi(\chi),\eta)$$
$$\mathbf{S}(\chi,\eta) \equiv \mathbf{R}[\psi(\chi,\eta)),\eta)]$$

The relation **T** can be reduced by negation to a case analogous to **S**, so that Proposition IV is proved.

Here, Gödel begins to imply that the numerical names assigned to the symbols and strings of symbols of the system **P** do not merely *correspond* to the respective symbols or strings of symbols to which they are assigned, but must actually be *equivalent* to them: to the extent that the ones may be substituted for the others.

We draw the reader's attention to the fact that for the above propositions to hold, the symbols 'χ', 'η', '**n**', '**n+1**', *etc.* used in them must represent concepts that *are defined in exactly the same way* on either side of the equation sign '=' or equivalence sign '≡' (as the case may be). Obviously there can be no equivalence between terms that are not defined equivalently — *i.e.*, represent the same concept.[44] Otherwise the above propositions — *viz.*, Gödel's **Propositions I-IV** — cannot hold!

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[44] This is so ancient a principle that it goes back all the way to the pre-Aristotelian Stoics. As DeLong states in his book *A Profile of Mathematical Logic*, "Since primary logic has its historical roots in the Stoic logicians, it is perhaps appropriate that we begin with an example from Chrysippus: '*Either the first or the second or the third / Not the first / Not the second / Therefore the third*.' What has been presented is not an argument but an argument form, a kind of logical skeleton. The occurrence of the phrases 'the first', 'the second', 'the third' is not necessary in order that this argument form be presented. By using *'A', 'B'* and *'C'* we can present the same form. Thus: *Either A or B or C / Not A / Not B / Therefore C.* We may turn an argument form into an argument by substituting sentences for the dummy letters. Our only requirement is that we substitute *the same sentence throughout for the same dummy letter*." (Emphasis added by us at the end to illustrate the principal error Gödel has made in his 1931 Paper.)

However, in *(xvii)* and *(xviii)* above, Gödel has defined symbols such as '**1**', '**3**', '**13**' and '$\mathbf{2^{11} \cdot 3^1 \cdot 5^{13}}$' as *designators* or *indicators* or *representatives* of symbols and of strings of symbols (*i.e.*, of formulae and proofs) of the system **P**. As such they belong to the metalanguage **G**. But as we saw in of **PART 1-B** of our Critique (*q.v.*), these definitions are incompatible with these same symbols being defined *also* as indicators or representatives of natural numbers — or of variables capable of being substituted by natural numbers. Such terms belong to the object-language **P**, and thus for the purposes of establishing a sound metalanguage proof may *not* be considered to be equivalent to the symbols belonging to the metalanguage **G** … even though they may *look* the same.

And as we showed from the example given above from the Stoic logician Chrysippus, to use the same symbol to represent two different concepts[45] in a single argument (or equation, or definition) is to commit a logical error of the most basic kind, resulting in the possibility of contradictions arising therefrom.

(xxiii) However, this is precisely the error Gödel makes in some — though admittedly not all — of his following *soi-disant* "series of functions (and relations) 1-45":

> The functions **x+y**, **x.y**, $\mathbf{x^y}$, and also the relations **x** < **y**, **x = y** are readily found to be recursive; starting from these concepts, we now define a series of functions (and relations) 1-45, of which each is defined from the earlier ones by means of the operations named in Propositions I to IV. This procedure, generally speaking, puts together many of the definition steps permitted by Propositions I to IV. Each of the functions (relations) 1-45, containing, for example, the concepts "FORMULA", "AXIOM", and "IMMEDIATE CONSEQUENCE", is therefore recursive.

Now this last statement is just not true. As we saw in *(xxi)* above, for any definition of "recursive" to hold, the symbols '$\mathbf{x_1}$', '$\mathbf{x_2}$', '$\mathbf{x_n}$' and '**k**' on each side of the two equations of Gödel's expression (2), namely

$$\phi(0,x_2,\ldots x_n) = \psi(x_2,\ldots x_n)$$
$$\phi(k+1,x_2,\ldots x_n) = \mu(k,\phi(k,x_2,\ldots x_n),x_2,\ldots x_n). \qquad (2)$$

… and on the two sides of the equivalence sign '≡' in his expression

$$R(x1,x2,\ldots xn) \equiv [\phi(x1,x2,\ldots xn) = 0]$$

[45] Of course if the two different concepts are such that each can be derived from the other using the axioms and rules of inference of the particular language in which they are defined, there is no error. Thus, for example, one may define, in the system **P**, the symbol '**3**' to be equivalent to the series of symbols '**1 + 2**', as well as define it to be equivalent to the series of symbols '**1 + 1 + 1**'. These two definitions are logically equivalent to one another; they can be demonstrated to be so using the axioms and rules of inference of the system **P**. Therefore the two above definitions are not logically two separate ones. (This is similar to the way the natural number **11** is also the prime number **11**, and the even number **2** is also the natural number **2** … as we saw earlier). However, if in Gödel's argument — *i.e.*, in the metalanguage **G** — one defines the symbol '**3**' to be (*i*) equivalent to the symbol '***f***' of the system **P** as well as (*ii*) equivalent to the symbols '**1 + 2**' of the system **P** — as Gödel actually does — then since these two definitions are *not* equivalent to one another, defining the symbol '**3**' in this fashion leads to a logical error.

… must be defined in the same way — that is to say, in other words, that each and every such symbol must denote the same thing on both sides of the equality sign '=' or the equivalence sign '≡' (as the case may be).

But if Gödel wishes to define the concepts "FORMULA", "AXIOM", and "IMMEDIATE CONSEQUENCE", as recursive, he has to define these terms using *two* incompatible definitions: *(i)* as a "natural number", defined in exactly the same way the term "natural number" is defined in the mathematical interpretation of the system **P**, and therefore belonging to the object-language **P**, and *(ii)* as a designator of one or more symbols — or series of symbols — of the system **P**, and therefore belonging to the metalanguage **G**. Since the two definitions *(i)* and *(ii)* above belong to two different languages, they may not be used in any single formula.

GÖDEL'S "FUNCTIONS (AND RELATIONS) 1-45"

Now we shall examine each of Gödel's definitions of his *soi-disant* "series of functions (and relations) 1-45" one by one, and see whether they all hold up. As we shall see, most of them do not.

We note, moreover, that one clear objective of Gödel's *soi-disant* "series of functions (and relations) 1-46" is the definition of his term '**Bew**', which according to him means "PROVABLE". Without these definitions Gödel cannot claim that the concept of "PROVABLE" is definable in the system **P**. However, as we have seen and as we shall see from what follows, in order to do this he has to define certain symbols, such as **x**, **y**, **z**, **n**, *etc.*, each in two different ways: as a sign or series of signs of the system **P** — which by definition belongs to the object-language — *and* as a numerical name or designator *of* a sign (or series of signs) of the system **P**, which by definition belongs to the metalanguage **G**. This is like defining *'apple'* as both a fruit *and* as a word, and renders virtually all his following definitions 1. to 46.[46] syncretic. *I.e.*, they lead to contradictions of the kinds explained for his definitions 9. and 10. in *(xxxii)* below (*q.v.*), and for his definition 15. in *(xxxvii)* below (*q.v.*).

Caveat: For each of the definitions 1. to 46. that follow, it must be noted that all the criticisms that apply to terms in the definiens apply to those in the definiendum too; and since (as Gödel says) "each is defined from the earlier ones by means of the operations named in Propositions I to IV", the criticisms that apply to the earlier ones apply to the later ones too.[47]

(xxiv) To begin with, Gödel defines the concept "**x** is divisible by **y**" as follows:

[46] Gödel actually defines 46 "functions (relations)", but the 46th one is claimed to be different from the others, in that he claims it not to be recursive.

[47] It is also to be borne in mind that since — as Gödel says — "each is defined from the earlier ones by means of the operations named in Propositions I to IV", if even *one* of the earlier definitions does not hold up, all those that follow would not hold up either.

1.

x/y ≡ (∃z)[z ≤ x & x = y.z]

x is divisible by **y**.

This statement is logically sound — *provided*, of course, that each of the terms **x**, **y** and **z** represents a natural number, or (more accurately) a variable which may be substituted by a natural number: and not a label applied to a basic symbol, or series of basic symbols, of the system **P**. That is to say, these symbols must all belong to the object-language **P** and not to the metalanguage **G**.

To illustrate: if they are considered as belonging to the object-language **P** — for example, if **z** is defined as a variable which may be substituted by the natural number **5**, **y** defined as a variable which may be substituted by the natural number **3**, and **x** defined as a variable which may be substituted by the natural number **9** — then the string of symbols **x/y** in the definiendum represents **9/3**, and the string **z ≤ x** in the definiens represents "**5** is less than or equal to **9**": both of which are correct (and "well-formed") expressions of the system **P**.

If however each of the terms '**x**', '**y**' and '**z**' is to be considered as belonging to the metalanguage **G**, *i.e.*, representing one or more basic symbols (or one or more series of basic symbols) of the system **P**, then the *soi-disant* function (or relation) "**x** is divisible by **y**" cannot hold — and of course, neither can the function (or relation) '**z ≤ x**'.

To give an example: if the symbols '**x**', '**y**' and '**z**' are considered as belonging to the metalanguage **G** such that '**x**' represents the symbol ***f***, the '**y**' the symbol ∀ and '**z**' the symbol ~ of the system **P**, then the string of symbols '**x/y**' certainly does not represent the statement "**x** is divisible by **y**", nor does the string of symbols '**z ≤ x**' represent "**z** is less than or equal to **x**". Under such definitions, '**x/y**' represents merely the string of symbols ***f*/∀**, and '**z ≤ x**' represents ~ ≤ ∀ … neither of which means anything at all in *either* the system **P** — regardless of the interpretation given to it — *or* in the metalanguage **G**.

In other words, they are not what Gödel would call "meaningful", or what in our days we would call "well-formed".

Similarly, if '**x**' and '**y**' are metalanguage symbols representing FORMULAE — *i.e.*, a series of basic signs — of the system **P**, '**x**' and '**y**' cannot be "divisible" by one another: a formula *per se* is not capable of being divisible by another.

Thus if '**x**' and '**y**' here are to be taken as belonging to the metalanguage **G**, each of them may *not* represent a natural number "assigned" to a basic sign or formula of the system **P**[48] — at least not if they are being used to establish a metamathematical proof.

[48] Unless in such a formula, those symbols are natural numbers themselves — but even then, if the natural number to which '**x**' corresponds is not divisible by the natural number to which '**y**' corresponds, it cannot be said that "**x** is divisible by **y**".

And if '**x**' and '**y**' *cannot* each represent a variable which may be substituted by a natural number, Gödel's definition 1. cannot be *recursive:* for in order for it to be recursive according to the definition of "recursive" given by Gödel, '**x**' and '**y**' *must* represent variables which may be substituted by natural numbers, and *nothing else*.[49]

(xxv) Now Gödel defines the concept "prime number" as follows:

2.

Prim(x) ≡ ~(∃z)[z ≤ x & z ≠ 1 & z ≠ x & x/z] & x > 1

x is a prime number.

Here too, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above apply (with the understanding, of course, that there is no symbol '**y**' being considered here). And as a result, here too, '**x**' being a metalanguage symbol, it may not validly represent a variable which may be substituted by a natural number "assigned" to a basic sign or formula of the system **P** — at least not if they are being used to establish a metamathematical proof.

(xxvi) Now Gödel defines the "**n**-th (in order of magnitude) prime number contained in **x**" as follows:

3.

0 Pr x ≡ 0
(n+1) Pr x ≡ εy [y ≤ x & Prim(y) & x/y & y > n Pr x]

n Pr x is the **n**-th (in order of magnitude) prime number contained in **x**.

Here too, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above apply (with the understanding, of course, that there is no symbol '**z**' being considered here). And as a result, here too neither '**x**' and '**y**' here can represent a variable which may be substituted by a natural number "assigned" to a basic sign or formula of the system **P** — at least not if they are being used to establish a metamathematical proof.

(xxvii) Now Gödel defines the concept of "factorial":

4.

0! ≡ 1
(n+1)! ≡ (n+1).n!

Again, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above apply, with the symbol '**n**' used instead of the symbols '**x**', '**y**' or '**z**'. And as a result, here too, '**n**' *cannot* represent a vari-

[49] Not that it matters too much, however, for we shall prove that Gödel's Theorem cannot hold even if it is assumed that all his definitions 1. to 45. *are* recursive.

able which may be substituted by a natural number "assigned" to a basic sign or formula of the system **P.**

(xxviii) Here Gödel defines the "**n**-th prime number (in order of magnitude)":

5.

Pr(0) ≡ 0
Pr(n+1) ≡ εy [y ≤ {Pr(n)}! + 1 & Prim(y) & y > Pr(n)]

Pr(n) is the **n**-th prime number (in order of magnitude).

Once again, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above apply, with the symbol '**n**' used instead of the symbols '**x**' or '**z**'. And as a result, here too, neither '**n**' nor '**y**' can represent a variable which may be substituted by a natural number "assigned" to a basic sign or formula of the system **P**.

(xxix) At this point in his definitions, more serious errors in Gödel's argument begin to reveal themselves to the critical mind.

Gödel defines his term '**n Gl x**' as follows:

6.

n Gl x ≡ εy [y ≤ x & x/(n Pr x)^{y} & not x/(n Pr x)$^{y+1}$]

n Gl x is the **n**-th term of the series of numbers assigned to the number **x** (for **n** > 0 and **n** not greater than the length of this series).

Now according to Gödel's functions / relations 1. to 5. above, and his definition of 'ε**x F(x)**' in *(xxii)* above, the definition of '**n Gl x**' in 6. above must *also* mean:

"**n Gl x** is equivalent to the smallest number **y** for which the following holds: **y** is equal to or less than **x**, and **x** is divisible by the **n**-th prime number contained in **x** when that **n**-th prime number is raised to the power of **y**, and **x** is not divisible by the **n**-th prime number contained in **x** when that **n**-th prime number is raised to the power of **y+1**".

Thus if the two definitions of the term '**n Gl x**', namely:

(a) the smallest number **y** for which the following holds: **y** is equal to or less than **x**, and **x** is divisible by the **n**-th prime number contained in **x** when that **n**-th prime number is raised to the power of **y**, and **x** is not divisible by the **n**-th prime number contained in **x** when that **n**-th prime number is raised to the power of **y+1**

… and

(b) the **n**-th term of the series of numbers assigned to the number **x** (for **n** > 0 and **n** not greater than the length of this series)

… are to be considered *equivalent* to each other, then the definitions of the terms '**x**', '**y**' and '**n**' contained in one of them must be equivalent to the definitions of the terms '**x**', '**y**' and '**n**' contained in the other.

But this, as we have seen, is not the case: the definition of '**x**' and '**y**' in *(a)* above — if Gödel's function/relation 6. is to be considered recursive — must be that of variables belonging to the object-language **P** — *i.e.*, each of which may be substituted by a natural number as "natural number" is defined in the mathematical interpretation of the system **P**; while the definition of '**n**' in *(b)* above — although not spelled out explicitly — is implied to be[50] that of the *numerical name* (or *designator* or *indicator*) of *a symbol* or a *series of symbols* of the system **P**.

In other words, '**x**' and '**y**' belong to the object-language **P**, while '**n**' belongs to the metalanguage **G**. This would readily have been seen had Gödel written his definition correctly, *i.e.*, distinguishing metalanguage terms from object-language terms, for instance by printing them in a different font, as follows:

$$n\ Gl\ \mathbf{x} \equiv \varepsilon \mathbf{y}\ [\mathbf{y} \leq \mathbf{x}\ \&\ \mathbf{x}/(n\ Pr\ \mathbf{x})^{\mathbf{y}}\ \&\ not\ \mathbf{x}/(n\ Pr\ \mathbf{x})^{\mathbf{y+1}}]$$

When written thus, it is easy to see that the above definition combines object-language and metalanguage terms in one single formula, and is therefore syncretic. Such a definition would be like considering *'apples'* the word as belonging to the same set as *lemons* the fruits. And as we showed conclusively in **PART 1** of our Critique, such a procedure would lead to the possibility of deriving contradictions from Gödel's definitions and arguments.

Thus, also, Gödel's definition 6. cannot be recursive.

Note also that if there is *any* "series of numbers assigned to **x**", then by implication, from what Gödel writes in *(xviii)* earlier, *viz.*,

> A natural number is thereby assigned in one-to-one correspondence, not only to every basic sign, but also to every finite series of such signs.

… '**x**' must be *represent* — and not merely *correspond to* — a basic sign or series of such signs of the system **P.**

Moreover, here too, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above (*q.v.*) apply — with the understanding, of course, that there is no symbol '**y**' being considered here. And as a result of these arguments, here too, '**x**' *cannot* represent a variable which may be substituted by a natural number which merely "corresponds" to a basic sign or formula of the system **P** — contrary to Gödel's words in *(xviii)* above and *(xvii)* earlier, *viz.,*

> … to every finite series of basic signs (and so also to every formula) there corresponds, one-to-one, a finite series of natural numbers.

Thereby Gödel by implication contradicts his explicitly-stated words.

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[50] Without such an implication, of course, there cannot *be* a "series of numbers assigned to the number **x**".

(xxx) Now we come to the definition of what Gödel calls a "function (relation)" which seems, on the face of it, quite absurdly defined. Observe the discrepancy between Gödel's interpretation of the function **l(x) ≡ εy [y ≤ x & y Pr x > 0 & (y+1) Pr x = 0]** — namely "**l(x)** is the length of the series of numbers assigned to **x**" — and the same function when it is interpreted according to Gödel's functions / relations 1. to 5. above, and his definition of '**εx F(x)**' in *(xxii)* above:

7.

l(x) ≡ εy [y ≤ x & y Pr x > 0 & (y+1) Pr x = 0]

l(x) is the length of the series of numbers assigned to **x**.

Note that according to Gödel's functions / relations 1. to 5. above and his definition of '**εx F(x)**' in *(xxii)* above, the definition of '**l(x)**' in 7. above must mean:

> "**l(x)** is equivalent to the smallest number **y** for which the following holds: **y** is equal to or less than **x**, the **y**-th prime number contained in **x** is greater than **0**, and the **(y+1)**-th prime number contained in **x** is equal to **0**."

This is obviously impossible, for if the **y**-th prime number contained in **x** is *greater than* **0**, then the **(y+1)**-th prime number contained in **x** cannot possibly be *equal to* **0**. (Perhaps this is a typographical error.)[51]

Whatever the case — it is certainly forgivable for Gödel to have made a few typographical errors in his document, if that is what this is — the same criticism as in *(xxix)* above applies: namely, that the two definitions of the term '**l(x)**', *viz.*,

(a) **l(x)** is the length of the series of numbers assigned to **x**

… and

(b) **l(x)** is equivalent to the smallest number **y** for which the following holds: **y** is equal to or less than **x**, the **y**-th prime number contained in **x** is greater than **0**, and the **(y + 1)**-th prime number contained in **x** is equal to **0**

… can only be equivalent to each other if the definitions of the terms '**x**' and '**n**' contained in one of them are equivalent to the definitions of the terms '**x**' and '**n**' contained in the other. This, as we have seen in *(xxix)* above, is just not the case. (*N.B.:* An '**n**' is not explicitly mentioned in *(a)* above, but here too — as in *(xxix)* above — an '**n**' is implied as denoting "the series of numbers assigned to **x**".)

And, of course, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above applies here too.

[51] Or perhaps what Gödel *intends* to say is that there *is* no **(y + 1)**-th prime number contained in **x** — that, in other words, **y** is the largest prime number contained in **x**.

(xxxi) The next function/relation Gödel defines is:

8.

x * y ≡ εz [z ≤ [Pr{l(x)+l(y)}]$^{x+y}$ & (n)[n ≤ l(x) → n Gl z = n Gl x] &
(n)[0 < n ≤ l(y) → {n+l(x)} Gl z = n Gl y]]

x * y corresponds to the operation of "joining together" two finite series of numbers.

Now the definition of the term '**x * y**' is dependent on the interpretation of the term '**l(x)**', which as we have seen is not altogether clear. But if we assume — to be generous toward Gödel — that the correct definition of '**l(x)**' is the one he has explicitly given in *(xxx) (a)*, then the definition of '**x * y**' ought to be as follows:

> "**x * y** is equivalent to the smallest number **z** for which the following holds: **z** is less than or equal to the (**x+y**)-th prime number raised to the power of **x+y**, and for all **n**, **n** is such that it satisfies the conditions that **n** is less than or equal to length of the series of numbers assigned to **x** which implies that the **n**-th term of the series of numbers assigned to the number **z** is equal to **n**-th term of the series of numbers assigned to the number **x** (for **n** > 0 and **n** not greater than the length of this series), and for all **n**, **n** is such that it satisfies the conditions that **n** is greater than **0** but less than or equal to the length of the series of numbers assigned to **y** which implies that the **n+l(x)**-th term of the series of numbers assigned to **z** is equal to the **n**-th term of the series of numbers assigned to **y**."

It is not necessary, of course, to actually try and understand or make sense of the above complicated definition, but it *is* necessary to realise that the above definition — which is a definition belonging to Gödel's metalanguage **G** — only holds true if the terms '**x**', '**y**', '**z**' and '**n**' are each defined in it as a variable which may be substituted by a natural number belonging to the object-language **P** (*i.e.,* as "natural numbers" are defined in the mathematical interpretation of the system **P**), and *in no other way*. As we see from *(xvii)* above, however, this is just not the case.

And, of course, the same caveat as in *(xxiii)* above applies here too.

(xxxii) The next two terms Gödel defines are as follows:

9.

R(x) ≡ 2^{x}

R(x) corresponds to the number-series consisting only of the number **x** (for **x** > 0).

10.

E(x) ≡ R(11) * x * R(13)

E(x) corresponds to the **operation** of "bracketing" [11 and 13 are assigned to the basic signs "**(**" and "**)**"].

Now some contradictions arising from the syncretism implicit in the process of "Gödelisation" become readily apparent. Observe: if we take definition 9. at face value, then it should be permissible to substitute in definition 10. above the term '$\mathbf{2^{11}}$' for the term '**R(11)**', because by definition 9., $\mathbf{R(x) \equiv 2^x}$; and for the same reason, it should be permissible to substitute '$\mathbf{2^{13}}$' for '**R(13)**'. In that case, we get the definition:

10a.

$$\mathbf{E(x) \equiv 2^{11} * x * 2^{13}}$$

And if, as Gödel says,

> **E(x)** corresponds to the operation of "bracketing",

… then the following equivalence must hold:

10b.

$$\mathbf{2^{11} * x * 2^{13} \equiv (x)}$$

From this it is very clear that the definiens in 9. above, namely '$\mathbf{2^x}$', is syncretic: that is to say, it combines two distinct definitions in one singe term. The symbol **2** in '$\mathbf{2^x}$' belongs to the object-language, for it represents a natural number (as that term is defined in the mathematical interpretation of the system **P**); while the '**x**' in '$\mathbf{2^x}$' belongs to Gödel's metalanguage **G**, wherein it designates or represents a basic sign or series of basic signs of the object-language **P**.

It therefore must be asked how the natural number **2** can meaningfully be raised to the power of a basic sign or series of basic signs of the system **P**. Unless the term **x** represents a VARIABLE which may be substituted by a natural number — *i.e.*, a "VARIABLE of first type" as Gödel puts it — it cannot possibly be used to raise the natural number **2** to the power of anything whatsoever!

This would again have become clearly apparent had Gödel written his definition 9. correctly, *i.e.*, by separating the metalanguage terms from the object-language terms, again (for example) by printing them in a different font as follows:

$$\mathcal{R}(x) \equiv \mathbf{2}^{x}$$

When written thus it is clear that the symbol **2** belongs to the object-language while the symbols $\mathcal{R}$ and x belong to the metalanguage; and as a result, the term $\mathbf{2}^{x}$ is syncretic, and therefore cannot be validly used in a metamathematical proof.

Indeed the entire definition, as well as definition 10., are seen to be syncretic when thus written: they both combine terms from the object-language with terms from the metalanguage.

And as a result, of course, the definitions 9. and 10. cannot be recursive.

As we saw, the term **2^x** — taken by *itself* — is syncretic. This also means that it must be defined *simultaneously* in *both* the object-language **P** *and* the metalanguage **G** (for if that were not the case, then it would be, as Gödel puts it, "not meaningful").

However, if the single term **2^x** is defined *simultaneously* in *both* the object-language **P** *and* the metalanguage **G**, it cannot be written in two separate fonts, or in any other way which distinguishes between object-language and metalanguage: for no such distinction would then exist.

But in that case it can lead to incorrect and absurd conclusions. For suppose — just to give an example — that **x** represents a VARIABLE of first type, *i.e.*, for natural numbers; and that we substitute the variable **x** by the natural number **10** (as the natural number **10** is defined in the mathematical interpretation of the object-language **P**). Then **$2^{11} * x * 2^{13} \equiv (x)$** becomes equivalent to:

10c.

$$2^{11} * 10 * 2^{13} \equiv (10)$$

Dividing the definiens of the above equivalence by **2** (as is allowed in the mathematical interpretation of the object-language **P**), we get

10d.

$$(10)/2 \equiv (5)$$

… whereas dividing the definiendum of the equivalence by **2** we get

10e.

$$(2^{11} * 10 * 2^{13})/2 \equiv (2^{10} * 5 * 2^{12})$$

But the right side of 10e. means *nothing at all* in the metalanguage **G**, while in the object-language **P** it is supposed to be — according to Gödel — equivalent to "joining together" the natural numbers **2^{10}**, **5**, and **2^{12}**. Since in the mathematical interpretation of the object language **P** the natural number **2^{10}** is equal to **1024**, and the natural number **2^{12}** is equal to **4096**, this implies that in the object-language **P**, "joining together" the natural numbers **2^{10}**, **5**, and **2^{12}** results in "joining together" the natural numbers **1024**, **5** and **4096**.

Exactly how one is supposed to "join" these natural numbers together in the system **P** is not too clear from Gödel's words, but assuming that we are required to "join" them together thus:

10f.

$$(2^{11} * 10 * 2^{13})/2 \equiv (2048108192)/2 \equiv (102454096)$$

… *i.e.*, assuming that the symbol '*' corresponds to the juxtaposition of the terms on either side of it without any spaces or signs in between — which is what is implied by definition 10. above

— then by considering definitions 10d., 10e. and 10f. together,[52] we get in the object-language **P** the following "equivalence":

10g.

(5) ≡ (102454096)

… which, especially when considered in light of Gödel's definition of **(x)** as "the class whose only element is **x**" — see *(xiv)* above — means "the class whose only element is **5** is equivalent to the class whose only element is **102454096**" … and which in turn, under a consistent formal axiomatic system capable of defining natural numbers, such as the system **P**, is utterly and absolutely incorrect.[53]

The same applies if the joining together of these numbers is done via '+' (*i.e.*, addition) signs, or ' ' (*i.e.*, multiplication) signs, or any other signs which result in operations allowed within the limits of the axioms and rules of inference of system **P** to be carried out on natural numbers.

On the other hand, if there is another way Gödel envisages of "joining together the natural numbers **1024**, **5** and **4096**", then what can it be? Every other conceivable way — such as for example

10h.

(1024-5-4096)

… or

10i.

(1024 5 4096)

… or

10j.

(1024,5,4096)

… results in an expression which is not definable in the mathematical interpretation of the system **P**.

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[52] Gödel's original German word is *"Aneinanderfügens"* (quote marks included), which corresponds to *anfügen* meaning "to attach,", "to annex" or "to append"; and in *Kurt Gödel: Collected Works* published by Oxford University Press (1986), the translation of *"Aneinanderfügens"* is given as "concatenating". These facts also support our interpretation above.

[53] Of course **(5) ≡ (102454096)** would be utterly and absolutely incorrect no matter how **(x)** were defined.

The logical consequence of all the above is, that either the two expressions **R(x) ≡ 2^x^** and **E(x) ≡ R(11) * x * R(13)** lead to a false — or incorrect — conclusion (as defined by the axioms and rules of inference of the system **P**, or for that matter, of any other consistent formal axiomatic system capable of defining natural numbers); or else to an expression not definable in such a system.

(xxxiii) Gödel's next definition is:

11.

n Var x ≡ (∃z)[13 < z ≤ x & Prim(z) & x = z^n] & n ≠ 0

x is a VARIABLE of **n**-th type.[54]

The same argument applies here as it does in *(xxxii)* above: the expression **x = z^n** in the definiens above is syncretic. As a result, the definition cannot be recursive, and the expression **x = z^n** would lead either to an incorrect expression of the system **P** or to an expression which cannot be defined within the limits of the axioms and rules of inference of the system **P**; and consequently the metalanguage expression **n Var x** also cannot be definable in the system **P** (contrary to what is implied by Gödel having put the word "VARIABLE" in SMALL CAPS).

(xxxiv) Gödel continues:

12.

Var(x) ≡ (∃n)[n ≤ x & n Var x]

x is a VARIABLE.

Since the expression **n Var x** in the definiens above cannot be defined in the system **P**, neither can the expression **Var(x)**. And of course, since **x** and **n** are defined differently from one another — **x** in the metalanguage **G** and **n** in the object-language **P** — the definition is not recursive (*i.e.*, the same argument as in *(xxix)* above applies).

(xxxv) After this, Gödel says:

13.

Neg(x) ≡ R(5) * E(x)

Neg(x) is the NEGATION of **x**.

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[54] It should be noted that up till his definition 9. above, Gödel has used the symbols '**x**' and '**y**' to refer exclusively to natural numbers — *i.e.*, what he calls "variables of first type". Now he begins to use the symbol '**x**' to represent variables of *any* type. There is no consistency in his entire Paper as to what he intends the symbols '**x**', '**y**', '**z**', '**n**' *etc*. to represent from one expression to another. (This is only a minor stylistic defect, however, and need not be dwelt upon at any length.)

Here we find an attempt to separate the object-language **P** from the metalanguage **G**. In the system **P**, the negation of a proposition is expressed by the symbol '~' (*vide* page 93, Vol. I, of *Principia Mathematica*.) Thus, for instance, if ***p*** represents the elementary proposition "the number **n** belongs to the class **c**", then **~*p*** represents the proposition "it is false that the number **n** belongs to the class **c**". In Gödel's terminology, however, if **x** represents a FORMULA which, when interpreted as to content, states "the number **n** belongs to the class **c**", then it would be **Neg(x)** which, when interpreted as to content, states "the number **n** does not belong to the class **c**".

Thus the symbol '**Neg**' applies, presumably, to the metalanguage **G** while '~' applies to the object-language **P**.

However, Gödel does not keep separate the symbols '**Neg**' and '~'. At times he applies '**Neg**' to one of his metalanguage FORMULAE, while at other times, and to another metalanguage FORMULA, he applies '~'. In *practice*, therefore, there is no separation of the object-language **P** from the metalanguage **G**, particularly after his expression (8.1) which we shall be discussing in **PART 2-B** of our critique.

(xxxvi) Gödel's next definition is:

14.

x Dis y ≡ E(x) * R(7) * E(y)

x Dis y is the DISJUNCTION of **x** and **y**.

Once again we can observe the absurdity resulting from a confusion of object-language and metalanguage, as follows.

In the definiens of definition 14., of course, the symbol '**R(7)**' — which by definition 9. above is equivalent to $\mathbf{2^7}$, or **128** — can only represent a natural number: *i.e.*, a term belonging to the object-language **P**. Thus definition 14. here is equivalent to the following expression:

x Dis y ≡ E(x) * 128 * E(y)

Since by Gödel's definition 10., '**E(x)**' is equivalent to '**R(11) * x * R(13)**', and since '**R(11)**' is equivalent to $\mathbf{2^{11}}$ or **2048**, and '**R(13)**' is equivalent to $\mathbf{2^{13}}$ or **8192**, then assuming — as earlier in *(xxxii)* above — that the symbol '*****' corresponds to the juxtaposition of the terms on either side of it without any spaces or signs in between, '**R(11) * x * R(13)**' is equivalent to the term '**2048x8192**'. Similarly, '**E(y)**' is equivalent to **2048y8192**

Thus Gödel's equivalence can be expressed, according to his own definitions, as:

x Dis y ≡ 2048x81921282048y8192

The only way the definiens can be considered "meaningful" in the system **P** is if **x** and **y** are each defined as a variable which may be substituted by a natural number — in which case, the definiens means "the natural number **2048** multiplied by the natural number **x** multiplied by the natural number **81921282048** multiplied by the natural number **y** multiplied by the natural number **8192**".

Thus for example if **x=5** and **y=10**, the answer is **6.87206** multiplied by $\mathbf{10^{19}}$ — a rather large number no doubt, but still just a natural number.

Whereas if the expression **x Dis y** is to be interpreted, according to Gödel's words above, as "the DISJUNCTION of **x** and **y**", then according to his own words in *(xv)* earlier, this can also be expressed as **(x)**∨**(y)**, and if as above **x=5** and **y=10**, then (**x**)∨**(y)** ≡ **(5)**∨**(10)**.[55]

But according to Gödel's words in *(xiv)* earlier, **(5)** is defined as "the class whose only element is **5**", and **(10)** as "the class whose only element is **10**". Thus **(5)**∨**(10)** is defined as "the class whose only element is **5** or the class whose only element is **10**".

Within the limits of the system **P**, however, *there is no way* the expression "the class whose only element is **5** or the class whose only element is **10**" can be considered equivalent to the natural number **6.87206** multiplied by the natural number $\mathbf{10^{19}}$. In other words, it is incorrect to claim, within the limits set by the axioms and the rules of inference of the system **P** — and regardless of the interpretation given to the symbols of the system **P** — that that "the class whose only element is **5** or the class whose only element is **10** is equivalent to the natural number **6.87206** multiplied by $\mathbf{10^{19}}$."

Whereas if **x** and **y** are *not* defined as VARIABLES which may be substituted by natural numbers, the expression in the definiens above, namely **2048x81921282048y8192**, is "not meaningful" in the system **P** (*i.e.*, it is not a "well-formed" expression of the system **P**.)

Thus, the same criticism as in *(xxxii)* above applies: namely, the terms in the definiens result in a conclusion which — when derived within the limitations of the axioms and rules of inference of the system **P** — is incorrect, or else to an expression that is undefinable in the system **P**. As a result, the expression **x Dis y** is also not definable within the system **P**.

(xxxvii) Now we come to Gödel's most celebrated expression, **x Gen y**, from which his "undecidable" FORMULA **17 Gen r** is derived. (Note that he attempts to show, in his **Proposition VI**, that **17 Gen r** cannot be proved, nor can its NEGATION be proved, within the system **P**.) He says:

15.

x Gen y ≡ R(x) * R(9) * E(y)

x Gen y is the GENERALIZATION of **y** by means of the variable **x** (assuming **x** is a variable).

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[55] It may be argued at this juncture that the term **x Dis y** belongs to the metalanguage **G** while the term **(a)**∨**(b)** belongs to the object-language **P**; and thus the two may *not* be considered equivalent. But such an argument, of course, confirms what we have been saying all along, namely that the vast majority of Gödel's definitions 1. to 46. and their derivations — including his celebrated "undecidable FORMULA" **17 Gen r** — cannot belong to the object-language **P** … and therefore his "undecidable" FORMULA **17 Gen r** cannot be undecidable *in* the system **P**.

Let us analyse the definiens in this definition carefully, term by term.

By Gödel's definition 9. above — and assuming that there is no separation of object-language terms from metalanguage terms, *i.e.*, the expression is to be read *exactly* as Gödel has written it — **R(x)** is equivalent to $\mathbf{2^x}$.

Thus, also, **R(9)** must be equivalent to $\mathbf{2^9}$, that is to say, to **512**.

And by his definition 10. earlier, **E(x)** is equivalent to **R(11) * x * R(13)**, which in turn means that **E(y)** must be equivalent to $\mathbf{2^{11} * y * 2^{13}}$ or **2048 * y * 8192**.

Assuming that the symbol '*' or "joining together" represents the operation of juxtaposing the terms on either side of it without any spaces or signs in between — which is what Gödel's definition 8. given earlier implies — then **2048 * y * 8192** must be equivalent to **2048y8192**.

Thus, "joining together" the terms **R(x)**, **R(9)** and **E(y)**, we get the following definition:

15a.

$$\mathbf{x\ Gen\ y \equiv R(x) * R(9) * E(y) \equiv 2^x5122048y8192}$$

Now *unless* **x** and **y** are *both* defined — as Gödel puts it — as "VARIABLES of first type (*i.e.*, for natural numbers)", the definiens above, namely '$\mathbf{2^x5122048y8192}$', is not definable in the system **P**.

Therefore if Gödel's expression **x Gen y** is to be definable at all in the system **P** — which as we shall see is absolutely necessary for Gödel to prove that his expression **17 Gen r** is undecidable *in* the system **P** — it becomes absolutely necessary to define **x** as a VARIABLE which represents a natural number:[56] *i.e.*, a VARIABLE OF FIRST TYPE.

But if **x Gen y** is to mean "the GENERALIZATION of **y** by means of a VARIABLE **x**" — as Gödel would have it — then if **x** is to represent a VARIABLE OF **n**-th TYPE, **y** must represent a VARIABLE OF **(n+1)**-th TYPE. In other words, it is impossible under those conditions for *both* **x** *and* **y** to represent VARIABLES OF FIRST TYPE (*i.e.*, for natural numbers).

Thus Gödel must choose which definition of **x Gen y** he will require us to accept: *(a)* a definition such that **x Gen y** represents the GENERALIZATION of **y** by means of the VARIABLE **x**, or *(b)* a definition such that **x Gen y** belongs to the system **P**.

For he cannot have both: because if he chooses *(a)* above, then since **x Gen y** cannot, under that choice, belong to the system **P**, neither can **17 Gen r**, which is of course a specific instance of **x Gen y**. In that case, **17 Gen r** cannot be undecidable *in* the system **P** — contrary to Gödel's final conclusion.

[56] Indeed, it must be defined as a variable which represents a natural number, *as a natural number is defined in the mathematical interpretation of the system* ***P***, and nowhere else … not even in *another* consistent formal axiomatic system capable of defining natural numbers.

But if Gödel chooses *(b)* above, then it is to be noted that he also defines, in *(xv)* earlier (*q.v.*), the expression **x∀(a)** as "a **generalization**[57] of **a**". (This, of course, is the definition of **x∀(a)** in the system **P**.) So if Gödel's words in his definition 15. here, *viz.* "**x Gen y** is the GENERALIZATION of **y** by means of the VARIABLE **x** (assuming **x** is a VARIABLE)", are to be consistent with his definition of **x∀(a)** as "a **generalization** of **a**" — which definition, as noted above, belongs to the system **P** — then **x Gen y** must be equivalent to **x∀(y)**.

Now **x∀(y)** is defined, according to the process of "Gödel-numbering" or "Gödelisation", as **x * 9 * 11 * y * 13** or **x·911·y·13**. Working this multiplication out, this comes to **11843xy**. Thus we have these two definitions, 15b. and 15c.:

15b.

$$\mathbf{x\ Gen\ y \equiv 2^x 5122048y8192}$$

… and

15c.

$$\mathbf{x\ Gen\ y \equiv 11843xy}$$

… which results in the following equivalence:

15d.

$$\mathbf{x5122048y8192 \equiv 11843xy}$$

… and which, of course, only holds —if it can hold at all — when **x** and **y** are all defined as VARIABLES OF FIRST TYPE, *i.e.*, VARIABLES representing natural numbers as the term "natural number" is defined within the mathematical interpretation of the system **P**.

As a consequence, when **x** and **y** are replaced by natural numbers, **11843xy** must also be defined as a natural number.

On the other hand $\mathbf{2^x 5122048y8192}$ can be expressed as **41959800000** multiplied by **y** multiplied by $\mathbf{2^x}$ or $\mathbf{41959800000y{\cdot}2^x}$. So it is absolutely necessary for the following equivalence to hold:

15e.

$$\mathbf{1959800000y{\cdot}2^x \equiv 11843xy}$$

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[57] It is to be noted that Gödel uses **bold** type in this case, not SMALL CAPS. (In his original German text he uses *italics* instead of **bold** type, but since in Meltzer's translation (which we are using), all his italics have been put into **bold** type instead, so as to allow us to use *italics* in the usual way for emphasis, we shall retain Meltzer's convention of using **bold** type in place of Gödel's original German *italics*.)

But there can be absolutely *no* values of **x** and **y** which will satisfy the above equivalence (which, if **x** and **y** are to represent natural numbers, is also an equation) *and* satisfy the requirement that **x** and **y** represent VARIABLES OF FIRST TYPE (*i.e.*, for natural numbers.)[58]

Thus 15e. above is not correct, and when correctly expressed it must be written as follows:

15f.

$$41959800000y \cdot 2^x \neq 11843xy$$

As a consequence — since both $41959800000y \cdot 2^x$ and **11843xy** must be equivalent to **x Gen y** — we get the following:

15g.

x Gen y ≠ x Gen y

… which renders '**x Gen y**' self-contradictory.

And of course, if **x Gen y** is self-contradictory, then Gödel's celebrated **17 Gen r** — which is an instance of **x Gen y** — must also be self-contradictory.

And this in turn means that his "proof" must be invalid and that his "Theorem" cannot hold in a consistent system of bivalent logic such as the system **P** — just as the classical "Liar Paradox" cannot hold therein, either.[59]

We see from the above that if there is no separation between object-language terms and metalanguage terms, Gödel's definition of **x Gen y** is self-contradictory. And yet, when a sepa-

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[58] Both sides of the equation can be divided by **y**, and that eliminates the variable **y**, leaving only the variable **x** on each side, along with the constant. If **x** is a variable which should be substituted by a natural number, it can only take on integer values: *i.e.,* **x** can only be 1, or 2, or 3, *etc.* Now no matter what value **x** takes, as long as **x** is a natural number — *i.e.,* as long as **x** is a positive integer — with increasing **x** the left side increases in magnitude compared with than the right side, since when $x \geq 1$, 2^x is always greater than **x**. This is because the difference between 2^z and $2^{(x+1)}$ is always greater than the difference between **(x+1)** and **x**. (As you can see, $2^{(x+1)} = 2^x \cdot 2 = 2^x + 2^x$, and so while the difference between **(x+1)** and **x** is **1**, the difference between $2^{(x+1)}$ and 2^x is 2^x, which for all positive **x** is always greater than **1**: even if $x = 0, 2^x = 1$.) Since the left side is multiplied by 2^x while the right side is only multiplied by **x**, and since the left side is greater than the right side to begin with, the difference between them can only get larger and larger. Only if **y** is zero can the two sides both equal zero — but then zero is not a Gödel-number, *i.e.,* it does not represent *any* sign or string of signs of the system **P**. So neither **x** nor **y** in **x Gen y** can be zero.

[59] What Gödel *intends* to do — *vide* his **Proposition VI** — is to show that a FORMULA of the type **v Gen r** *asserts its own unprovability;* what he actually *succeeds* in doing, however, is merely showing that such a FORMULA *asserts its own falsehood* — *i.e.*, he succeeds only in reiterating the classical "Liar Paradox". Thus all the criticisms levelled against the Liar Paradox — such as the non-permissibility of a statement to state itself — apply to Gödel's "Theorem" too. Note that if one accepts the classical Liar Paradox, one is forced also to conclude that every bivalent system of formal logic must be *in*consistent: since if that were truly the case, a statement that asserts its own falsehood — *i.e.*, a statement that denies itself, in the form $p \equiv \sim p$ — could always be asserted in such a system. (Since $p \equiv p$ is always true, if $p \equiv \sim p$ is also asserted as being true, then both $p \equiv \sim p$ and $p \equiv p$ together end up reading $p \,\&\, \sim p$, which is the very epitome of inconsistency.)

ration between object-language terms and metalanguage terms *is* attempted for this definition, we get a completely syncretic formula — due to the syncretic nature of the term **R(x)** —which cannot possibly be used to soundly establish a metamathematical proof, because it attempts to *use* the terms *of* the object-language *in* the metalanguage.

(xxxviii) Gödel now continues:

16.

0 N x ≡ x

(n+1) N x ≡ R(3) * n N x

n N x corresponds to the operation: "**n**-fold prefixing of the sign '***f***' before **x**."

There is nothing much wrong with this definition … with the proviso, of course, that both **n** and **x** must represent VARIABLES OF FIRST TYPE, which may be substituted by natural numbers (as "natural numbers" are defined in the system **P**) — since otherwise Gödel's words above, *viz.*:

n N x corresponds to the operation: "**n**-fold prefixing of the sign '***f***' before **x**."

… make no sense.[60] Consequently they cannot *also* be defined as terms representing basic signs or series of basic signs of the system **P** — *i.e.*, they cannot also belong to the metalanguage **G** — for as we saw earlier in sub-section **(B)** of **PART 1** of our Critique, that would be incompatible with **x** and **n** belonging to the object-language **P**.

(xxxix) Gödel continues:

17.

Z(n) ≡ n N [R(1)]

Z(n) is the NUMBER-SIGN for the number **n**.

There is nothing much wrong with this definition either, with the same proviso as in *(xxxvii)* above, wherein we saw that in the term **n N [R(1)]**, both **n** and **[R(1)]** must represent VARIABLES OF FIRST TYPE, which may be substituted by natural numbers (as "natural numbers" are defined in the mathematical interpretation of the system **P**).

This is of course also corroborated by Gödel's definition 9., according to which **[R(1)]** is defined as equivalent to $\mathbf{2^1}$, which in turn is, of course, equal to the natural number **2**.

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[60] For instance, if '**x**' represents the sign '~' of the system **P**, then the term '***ffff***~' — which is not definable in the stem **P** — makes no sense. Similarly, if **n** represents, for example, the sign '∀' of the system **P**, then the sentence "an '∀-fold' prefixing of the sign '***f***' before '**x**'" makes no sense.

Thus the term **n** in **Z(n)** must also belong to the object-language **P**, representing therein a natural number (as "natural numbers" are defined in the mathematical interpretation of the system **P**), and cannot belong to the metalanguage **G**, representing therein a basic sign or series of basic signs of the system **P** — *i.e.*, the criticism given in *(xxxii)* above applies here too.

(xL) Gödel now says:

18.

$Typ_1'(x) \equiv (\exists m,n)\{m,n \leq x \,\&\, [m = 1 \vee 1 \text{ Var } m] \,\&\, x = n \text{ N } [R(m)]\}$

x is a SIGN OF FIRST TYPE.

Here, since the term **1 Var m** appears in the definiens, the criticism of **n Var x** in *(xxxiii)* above applies as well. Also, the same remarks as in *(xxxvii)* above apply. (Note that earlier — see *(xv)* above — Gödel has defined a "sign of first type" as follows: "By a **sign of first type** we understand a combination of signs of the form: **a**, ***f*a**, ***ff*a**, ***fff*a** … etc. … where **a** is either **0** or a variable of first type.")

(xLi) Gödel continues:

19.

$Typ_n(x) \equiv [n = 1 \,\&\, Typ_1(x)] \vee [n > 1 \,\&\, (\exists v)\{v \leq x \,\&\, n \text{ Var } v \,\&\, x = R(v)\}]$

x is a SIGN of **n**-th TYPE.

Again, since the term **n Var v** appears in the definiens, the criticism of **n Var x** in *(xxxiii)* above applies here too. Also the same remarks as in *(xxxvii)* above apply.

(xLii) Next, Gödel says:

20.

$Elf(x) \equiv (\exists y,z,n)[y,z,n \leq x \,\&\, Typ_n(y) \,\&\, Typ_{n+1}(z) \,\&\, x = z * E(y)]$

x is an ELEMENTARY FORMULA.

Here we again have serious problems. The expression **y,z,n ≤ x** in the definiens above can only be what Gödel calls "meaningful" (or what we nowadays might call "well-formed") if all the terms in it — **x**, **y**, **z** and **n** — represent VARIABLES OF FIRST TYPE, which may be substituted by numbers (natural or otherwise). But **x** is also defined as an "ELEMENTARY FORMULA", which cannot be a number, natural or otherwise! (Note that earlier, *viz.* in *(xv)* above, Gödel has defined "elementary formulae" thus: "Combinations of signs of the form **a(b)**, where **b** is a sign of **n**-th and **a** a sign of **(n+1)**-th type, we call **elementary formulae**." This definition precludes any "ELEMENTARY FORMULA" from also being a number of any kind — natural or otherwise — especially since the fact that "ELEMENTARY FORMULA" is written in SMALL CAPS means that it belongs to the system **P**.)

Obviously, therefore, what Gödel *means to say* is that **x** belongs to the metalanguage **G**, wherein it represents the number "assigned" to an ELEMENTARY FORMULA for which the definiens holds. Which in turn implies that in his metalanguage **G** he must define **x** *both* as a natural number (*i.e.*, as "natural numbers" are defined in the mathematical interpretation of the system **P**), and *also* as a representative or designator (or identifier or numerical name) of the FORMULA of the system **P** which satisfies the definiens, *viz.*

$$(\exists y,z,n)[y,z,n \leq x \ \& \ Typ_n(y) \ \& \ Typ_{n+1}(z) \ \& \ x = z * E(y)].$$

But as we saw in sub-section **(B)** of **PART 1** of our Critique, these two definitions are mutually incompatible, and can result in contradictions, thus rendering Gödel's metalanguage **G** inconsistent.

(xLiii) Gödel now says:

21.

$$Op(x,y,z) \equiv x = Neg(y) \lor x = y \ Dis \ z \lor (\exists v)[v \leq x \ \& \ Var(v) \ \& \ x = v \ Gen \ y]$$

Since the expression **v Gen y** appears in the definiens, the criticism levelled by us against Gödel's celebrated expression **x Gen y** in *(xxxvii)* above (*q.v.*) apply to this definition too.[61]

(xLiv) Gödel continues:

22.

$$FR(x) \equiv (n)\{0 < n \leq l(x) \rightarrow Elf(n \ Gl \ x) \lor (\exists p,q)[0 < p,q < n \ \& \ Op(n \ Gl \ x, p \ Gl \ x, q \ Gl \ x)]\} \ \& \ l(x) > 0$$

x is a series of FORMULAE of which each is either an ELEMENTARY FORMULA or arises from those preceding by the operations of NEGATION, DISJUNCTION and GENERALIZATION.

Here too, criticisms similar to those enunciated in *(xL)* above apply. That is to say, since the term **n Gl x** appears in the definiens, the criticism of **n Gl x** in *(xxix)* above (*q.v.*) apply to the term **FR(x)** too. Besides, as we saw in *(xLii)* above, if **x** is to be defined in Gödel's metalanguage **G** as a "FORMULA" — *of* the system **P**, it goes without saying of course — then **x** cannot also be defined in the metalanguage **G** as a VARIABLE OF FIRST TYPE representing a natural number (as "natural number" is defined in the mathematical interpretation of the system **P**); and since **x** cannot be defined within the limits of the axioms and rules of inference of the system **P** as a "FORMULA" *of* the system **P**, the term **FR(x)** also cannot be defined within the limits of the axioms and rules of inference the system **P**.

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[61] The term '**Op**' is not defined by Gödel anywhere above. However, the translator has added a note to the effect that it stands for the German word *Operation* meaning (in English) "operation". (*Duh!*)

(xLv) Gödel continues:

23.

Form(x) ≡ (∃n){n ≤ (Pr[l(x)^2])$^{x.[l(x)]^2}$ & FR(n) & x = [l(n)] Gl n}

x is a FORMULA (i.e. last term of a SERIES OF FORMULAE **n).**

And here too, criticisms similar to those applied to *(xLiii)* and *(xLiv)* above (*q.v.*) apply — with the appropriate substitution of symbols of course. (We shall not repeat the criticisms since the appropriate substitutions can be made easily by the reader.)

(xLvi) Gödel continues:

24.

v Geb n,x ≡ Var(v) & Form(x) & (∃a,b,c)[a,b,c ≤ x & x = a * (v Gen b) * c & Form(b) & l(a)+1 ≤ n ≤ l(a)+l(v Gen b)]

The VARIABLE **v is** BOUND at the **n**-th place in **x.**

And here too, criticisms similar to those applied to *(xLiii)* and *(xLiv)* above (*q.v.*) apply. (Again, here and in the next few cases, we shall not repeat the criticisms, since the appropriate substitutions can be easily made by the reader.)

(xLvii) Gödel now says:

25.

v Fr n,x ≡ Var(v) & Form(x) & v = n Gl x & n ≤ l(x) & not(v Geb n,x)

The VARIABLE **v** is FREE at the **n**-th place in **x.**

Here too, criticisms similar to those applied to *(xLiii)* and *(xLiv)* above (*q.v.*) apply.

(xLviii) Gödel now says:

26.

v Fr x ≡ (∃n)[n ≤ l(x) & v Fr n,x]

v occurs in **x** as a free variable.

And here too, criticisms similar to those applied to *(xLiii)* and *(xLiv)* above (*q.v.*) apply.

(xLix) Gödel now says:

27.

Su x(n|y) ≡ εz {z ≤ [Pr(l(x)+l(y))]x+y &
[(∃u,v)u,v ≤ x & x = u * R(b Gl x) * v & z = u * y * v & n = l(u)+1]}

Su x(n|y) derives from **x** on substituting **y** in place of the **n**-th term of **x** (it being assumed that **0 < n ≤ l(x))**.

And here again, criticisms similar to those applied to *(xLiii)* and *(xLiv)* above (*q.v.*) apply.

(L) Gödel now defines two terms '**0 St v,x**' and '**(k+1) St v,x**' as follows:

28.

0 St v,x ≡ εn {n ≤ l(x) & v Fr n,x & not (∃p)[n < p ≤ l(x) & v Fr p,x]}
(k+1) St v,x ≡ εn {n < k St v,x & v Fr n,x & (_p)[n < p < k St v,x & v Fr p,x]}

k St v,x is the (**k+1**)-th place in **x** (numbering from the end of formula **x**) at which **v** is free in **x** (and **0**, if there is no such place.)

Here, **x** is explicitly asserted to represent a FORMULA of the system **P**, and thus must belong to the metalanguage **G**. As such, it obviously cannot represent a VARIABLE OF FIRST TYPE which may be substituted by a natural number (as "natural number" is defined in the mathematical interpretation of the object-language **P**.)

And thus here too, criticisms similar to those applied to *(xLiii)* and *(xLiv)* above (*q.v.*) apply.

(Li) Gödel now defines:

29.

A(v,x) ≡ εn {n ≤ l(x) & n St v = 0}

A(v,x) is the number of places at which **v** is FREE in **x**.

And here, once more, criticisms similar to those enunciated in *(xLiii)* and *(xLiv)* above (*q.v.*) must apply.

(Lii) Gödel now says:

30.

Sb_0(x v|y) ≡ x

Sb_{k+1}(x v|y) ≡ Su[Sb_k(x v|y)][(k St v, x)|y)]

Since the expressions **Su[Sb$_k$(x v|y)]** and **k St v, x** appear in the definiens, the criticisms applying to the terms **Su x(n|y)** in *(xLviii)* above and **k St v,x** in *(xLix)* above (*q.v.*) apply to this definition too.

(Liii) Gödel now says:

31.

Sb(xv|y) ≡ Sb$_{A(v,x)}$(x v|y)

Sb(x v|y) is the concept **Subst a(v|b)**, defined above.

Since the expression **Sb$_{A(v,x)}$(x v|y)** appears in the definiens, and since the expression **A(v,x)** appears in the definiens of **Sb(x v|y)** and the expressions **Su[Sb$_k$(x v|y)]** and **k St v, x** appear in the definiens of **Sb$_{k+1}$(x v|y)**, the criticisms applying to the term **A(v,x)** in *(Li)* above, the term **Su x(n|y)** in *(xLix)* above and the term **k St v,x** in *(xL)* above (*q.v.*) apply to this definition too.

(Liv) Gödel continues:

32.

x Imp y ≡ [Neg(x)] Dis y
x Con y ≡ Neg{[Neg(x)] Dis [Neg(y)]}
x Aeq y ≡ (x Imp y) Con (y Imp x)
v Ex y ≡ Neg{v Gen [Neg(y)]}

Since the terms **Neg(x)**, **[Neg(x)] Dis y** and **v Gen [Neg(y)]** appear in the definientia above, the criticisms applicable to the terms **Neg(x)**, **x Dis y** and **x Gen y** in *(xxxv)*, *(xxxvi)* and *(xxxvii)* above (*q.v.*) apply here too.

(Lv) Gödel now says:

33.

n Th x ≡ εn {y ≤ x$^{(x^n)}$ & (k) ≤ l(x) → (k Gl x ≤ 13 & k Gl y = k Gl x) ∨
(k Gl x > 13 & k Gl y = k Gl x.[1 Pr(k Gl x)]^n)]}

n Th x is the **n**-th TYPE-LIFT of **x** (in the case when **x** and **n Th x** are FORMULAE).

Since the terms **l(x)** and **k Gl x** appear in the definientia above, the criticisms applicable to the terms **l(x)**, **n Gl x** in *(xxix)* and *(xxx)* above (*q.v.*) apply here too. Moreover, here Gödel explicitly defines **x** as a "FORMULA" (*i.e.*, a FORMULA *of* the system **P**, obviously); and thus the expression **x$^{(x^n)}$** is utterly undefinable *in* the system **P**. Therefore the expression **n Th x** must also be utterly undefinable in the system **P**.

(Lvi) Gödel now continues:

To the axioms **I**, 1 to 3, there correspond three determinate numbers, which we denote by $\mathbf{z_1}$, $\mathbf{z_2}$, $\mathbf{z_3}$, and we define:

34.

Z–Ax(x) ≡ (x = $\mathbf{z_1}$ ∨ x = $\mathbf{z_2}$ ∨ x = $\mathbf{z_3}$)

Here too, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above apply (with the understanding, of course, that there is no symbol '**y**' being considered here).

(Lvii) Gödel now continues:

35.

$\mathbf{A_1}$-Ax(x) ≡ (∃y)[y ≤ x & Form(y) & x = (y Dis y) Imp y]

x is a FORMULA derived by substitution in the <u>axiom-schema II, 1</u>. Similarly **$\mathbf{A_2}$-Ax**, **$\mathbf{A_3}$-Ax**, **$\mathbf{A_4}$-Ax** are defined in accordance with the axioms **II**, 2 to 4.

Here too, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above apply. Moreover, since the terms **Form(y)** and **(y Dis y) Imp y** appear in the definiens, the criticisms applying to the term **Form(x)** in *(xLv)* above, the term **x Dis y** in *(xxxvi)* above and the term **x Imp y** in *(Liv)* above (*q.v.*) apply to this definition too.

(Lviii) Gödel now writes:

36.

A-Ax(x) ≡ $\mathbf{A_1}$-Ax(x) ∨ $\mathbf{A_2}$-Ax(x) ∨ $\mathbf{A_3}$-Ax(x) ∨ $\mathbf{A_4}$-Ax(x)

x is a FORMULA derived by substitution in an axiom of the sentential calculus.

Here too, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above apply (with the understanding, of course, that there is no symbol **y** being considered here).

(Lix) Gödel now defines:

37.

Q(z,y,v) ≡ (∃n,m,w)[n ≤ l(y) & m ≤ l(z) & w ≤ z & w = m Gl x & w Geb n,y & v Fr n,y]

z contains no VARIABLE BOUND in **y** at a position where **v** is FREE.

Here too, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above apply. Moreover, since the terms **l(y)** and **m Gl x** appear in the definiens, the criticisms applying to the term **l(x)** in *(xxx)* above and the term **n Gl x** in *(xxix)* above (*q.v.*) apply to this definition too.

(Lx) Gödel now defines:

38.

> **L_1-Ax(x) ≡ (∃v,y,z,n){v,y,z,n ≤ x & n Var v & Typn(z) & Form(y) & Q(z,y,v) & x = (v Gen y) Imp [Sb(v|z)]}**
>
> **x** is a FORMULA **derived** from the axiom-schema III, 1 by substitution.

Here too, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above apply. Moreover, since the terms **Form(y)** and **(v Gen y) Imp [Sb(v|z)]** appear in the definiens, the criticisms applying to the term **Form(x)** in *(xLv)* above, the term **x Gen y** in *(xxxvii)* above and the term **x Imp y** in *(Liv)* above (*q.v.*) apply to this definition too.

(Lxi) Gödel now says:

39.

> **L_2-Ax(x) ≡ (∃v,q,p){v,q,p ≤ x & Var(v) & Form(p) & v Fr p & Form(q) & x = [v Gen (p Dis q)] Imp [p Dis (v Gen q)]}**
>
> **x** is a FORMULA derived from the axiom-schema III, 2 by substitution.

Here too, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above apply. Also, since the terms **Form(p)**, **p Dis q**, **[v Gen (p Dis q)]** and **[v Gen (p Dis q)] Imp [p Dis (v Gen q)]** appear in the definiens, the criticisms applying to the term **Form(x)** in *(xLv)* above, the term **x Dis y** in *(xxxvi)* above, the term **x Gen y** in *(xxxvii)* above, and the term **x Imp y** in *(Liv)* above (*q.v.*) apply to this definition too.

(Lxii) Gödel now says:

40.

> **R-Ax(x) ≡ (∃u,v,y,n)[u, v, y, n ≤ x & n Var v & (n+1) Var u & u Fr y & Form(y) & x = u ∃x {v Gen [[R(u)*E(R(v))] Aeq y]}]**
>
> **x** is a FORMULA derived from the **axiom**-schema IV, 1 by substitution.

Here too, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above apply. Moreover, since the terms **u Fr y Form(y)** and **v Gen [[R(u)*E(R(v))] Aeq y]** appear in the definiens, the criticisms applying to the term **v Fr x** in *(xLviii)* above, the term **Form(x)** in *(xLv)* above and the term **x Gen y** in *(xxxvii)* above (*q.v.*) apply to this definition too.

(Lxiii) Gödel now continues:

> To the axiom V, 1 there corresponds a determinate number $\mathbf{z_4}$ and we define:

41.

M-Ax(x) ≡ (∃n)[n ≤ x & x = n Th z_4]

Here too, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above apply. Moreover, since the term **n Th z_4** appears in the definiens, the criticisms applying to the term **n Th x** in *(Lv)* above (*q.v.*) applies to this definition too.

(Lxiv) Gödel now continues:

42.

Ax(x) ≡ Z-Ax(x) ∨ A-Ax(x) ∨ L_1-Ax(x) ∨ L_2-Ax(x) ∨ R-Ax(x) ∨ M-Ax(x)

x is an AXIOM.

Here too, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above apply. Moreover, since the terms **Z-Ax(x) A-Ax(x)**, **L_{1-A}x(x), L_2-Ax(x)**, **R-Ax(x)** and **M-Ax(x)** appear in the definiens, the criticisms applying to these terms in *(Lvi)* to *(Lviii)* and *(Lx)* to *(Lxiii)* above (*q.v.*) apply to this definition too.

(Lxv) Gödel now continues**:**

43.

Fl(x y z) ≡ y = z Imp x ∨ (∃v)[v ≤ x & Var(v) & x = v Gen y]

x is an IMMEDIATE CONSEQUENCE of **y** and **z**.

Here too, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above apply. Moreover, since the terms **z Imp x**, **Var(v)** and **v Gen y** appear in the definiens, the criticisms applying to the term **x Imp y** in *(Liv)*, the term **Var(x)** in *(xxxiv)* and the term **x Gen y** in *(xxvxii)* above (*q.v.*) apply to this definition too.

(Lxvi) Gödel now continues:

44.

Bw(x) ≡ (n){0 < n ≤ l(x) → Ax(n Gl x) ∨ (∃p,q)[0 < p,q < n &
Fl(n Gl x, p Gl x, q Gl x)]} & l(x) > 0

x is a PROOF-SCHEMA (a finite series of FORMULAE, of which each is either an AXIOM or an IMMEDIATE CONSEQUENCE of two previous ones).

Here too, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above apply. Moreover, since the terms **Ax(n Gl x)**, **n Gl x** and **Fl(n Gl x, p Gl x, q Cl x)** appear in the definiens, the criticisms applying to the term **Ax(x)** in *(Lxiv)*, the term **n Gl x** in *(xxix)* and the term **Fl(x y z)** in *(Lxv)* above (*q.v.*) apply to this definition too.

(Lxvii-A) Gödel's next-to-last definition is:

45.

x B y ≡ Bw(x) & [l(x)] Gl x = y

x is a PROOF of the FORMULA **y**.

Here too, the same caveat as in *(xxiii)* and proviso as in *(xxiv)* above apply. Moreover, since the terms **Bw(x)** and **[l(x)] Gl x** appear in the definiens, the criticisms applying to the term **Bw(x)** in *(Lxv)* and the term **n Gl x** in *(xxix)* above (*q.v.*) apply to this definition too.

(Lxvii-B) Gödel's last definition is:

46.

Bew(x) ≡ (∃y)y B x[62]

x is a provable formula. (**Bew(x)** is the only one of the concepts 1-46 of which it cannot be asserted that it is recursive.)

Since the term **y B x** exists in the definiens, the criticism applicable to **x B y** in *(Lxv) above (q.v.)* applies to **Bew(x)** too.

Summing up, none of the above definitions can hold if — as implied by Gödel — the terms in them belong to both the object-language *and* the metalanguage. Defining metalanguage terms to be "materially equivalent to" or "the same as" object-language terms would be akin to defining the word *'apple'* as being the same as the fruit *apple.* We appreciate this very clearly when speaking of words as opposed to fruits, even when both are written with identical letters of the alphabet; it should, therefore, be only slightly more difficult — if at all — to appreciate this with respect to Gödel-numbers as opposed to natural numbers, even though both may be written with the help of identical digits.

As we shall see in **PART 2-B**, when no distinction is made between object-language symbols and metalanguage symbols, we can end up with contradictions similar to our earlier example of apples and lemons being words and not words simultaneously.

[62] In Meltzer's translation the sign used here is an equality sign ('='), but that must be a typographical error, because in Gödel's original German paper republished in *Kurt Gödel: Collected Works* (Oxford University Press) the equivalence sign ('≡') is used.

PART 2-B

We shall now observe the results of the syncretism pointed out by us in **PART 2-A**. As we shall see, the result is, that if there is *no* effort at separating object-language terms from metalanguage terms, self-contradictory FORMULAE can be enunciated — *i.e.*, Gödel's metalanguage **G** becomes inconsistent — or, if a separation *is* made between object-language terms and metalanguage terms, then it becomes impossible for Gödel's so-called "undecidable" FORMULA **17 Gen r** to belong to the object-language **P**.

And in either case, Gödel cannot prove his Theorem.

GÖDEL'S "PROPOSITION V"

(Lxviii) Gödel now continues:

> The following proposition is an exact expression of a fact which can be vaguely formulated in this way: every recursive relation is definable in the system **P** (interpreted as to content), regardless of what interpretation is given to the formulae of **P**:
>
> **Proposition V**: To every recursive relation $\mathbf{R(x_1 \dots x_n)}$ there corresponds an **n**-place RELATION-SIGN **r** (with the FREE VARIABLES $\mathbf{u_1, u_2, \dots u_n}$) such that for every **n**-tuple of numbers $\mathbf{(x_1 \dots x_n)}$ the following hold:
>
> $$\mathbf{R(x_1 \dots x_n) \rightarrow Bew\{Sb[r\ (u_1 \dots u_n)|(Z(x_1) \dots Z(x_n)]\}} \quad (3)$$
> $$\mathbf{\sim R(x_1 \dots x_n) \rightarrow Bew\{Neg\ Sb[r\ (u_1 \dots u_n)|(Z(x_1) \dots Z(x_n)]\}} \quad (4)$$
>
> We content ourselves here with indicating the proof of this proposition in outline, since it offers no difficulties of principle and is somewhat involved. We prove the proposition for all relations $\mathbf{R(x_1 \dots x_n)}$ of the form: $\mathbf{x_1 = \phi(x_2 \dots x_n)}$ (where ϕ is a recursive function) and apply mathematical induction on the degree of ϕ. For functions of the first degree (i.e. constants and the function **x+1**) the proposition is trivial. Let ϕ then be of degree **m**. It derives from functions of lower degree $\phi_1 \dots \phi_k$ by the operations of substitution or recursive definition. Since, by the inductive assumption, everything is already proved for $\phi_1 \dots \phi_k$, there exist corresponding relation-signs $\mathbf{r_1 \dots r_k}$ such that (3) and (4) hold. The processes of definition whereby ϕ is derived from $\phi_1 \dots \phi_k$ (substitution and recursive definition) can all be formally mapped in the system **P**. If this is done, we obtain from $\mathbf{r_1 \dots r_k}$ a new relation-sign **r**, for which we can readily prove the validity of (3) and (4) by use of the inductive assumption. A relation-sign **r**, assigned in this fashion to a recursive relation, will be called recursive.

(Lxviii-A) Note that Gödel does not give a *rigorous* proof of his **Proposition V**.

Now there are only two possibilities: either such a rigorous proof exists, or it doesn't. There is no other choice in bivalent logic; and bivalent logic is the only kind of logic that Gödel is using to attempt to prove his Theorem.[63]

Of course, if it be assumed that a proof of **Proposition V** does *not* exist, then we could conclude our critique right here, and roundly proclaim that Gödel has not proved his Theorem: for as can be seen from what follows, he absolutely and categorically *needs* **Proposition V** to even *attempt* to prove it.

But let us assume — to be generous toward Gödel — that a proof of **Proposition V** actually does exist. In that case, though, it can exist only in a way that *either* the antecedent and the consequent expression (3) hold, *or* the antecedent and the consequent of expression (4) hold. That is to say, expressions (3) and (4) are *conditionals*.

In the ordinary course of things, of course, this should not be a problem. But it is not permissible to make use of *both* the antecedents or *both* the consequents of expressions (3) and (4), or of any terms derived from them, in a *single* argument.

Moreover, it is also not permissible to use *only* the antecedent of one of these expressions, or *only* the consequent of one of these expressions, in order to prove anything. In a conditional formula, it is the formula itself that is correct; but that can be the case even if both the antecedent and the consequent are false (or incorrect).[64]

However, as we shall see, Gödel derives the consequents of his expressions (9) and (10) from the consequents of both his expressions (3) and (4), and the consequents of his expressions (15) and (16) from the consequents of his expressions (9) and (10). And then he uses *both* the consequents of expressions (15) and (16) in his natural language argument to show that his celebrated FORMULA **17 Gen r** is not PROVABLE in a class κ of FORMULAE, and that neither is its NEGATION PROVABLE in that same class κ.

This, however, would result in a contradiction — as we shall see. (The contradiction would not, of course, be a formal contradiction — *i.e.*, a contradiction arising from the mere form of the FORMULAE concerned, regardless of their interpretation — because here we are discussing the metalanguage and not the object-language. The metalanguage has to be *about* the object-language, and thus by definition cannot be un-interpreted.)

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[63] If one argues — as has been done — that Gödel uses Intuitionistic logic exclusively to attempt to prove his Theorem, then there *may* in fact be a third choice: that a proof of **Proposition V** does not *not* exist. (Intuitionistic logic does not accept the law of the excluded middle as being *always* valid, and so the double negation does not *necessarily* collapse.) However, in the above case, even Intuitionists would accept the notion that the law of excluded middle does hold; for according to the Intuitionistic philosophy, it does not hold only when infinities are brought into the picture — which is not the case here. And moreover, there can be no restriction in using the excluded middle to *refute* Gödel; for if there were, it would mean that classicists — who accept the law of the excluded middle as being *always* true — would never be able to accept Gödel's Theorem as valid!).

[64] Take for example the following conditional statement: "If there were world peace there would be no military casualties." Although world peace has never broken out and isn't likely to do so any time soon, the statement itself could be true even though neither its antecedent nor its consequent has ever been true.

And in addition, as we shall also see, Gödel makes use of *only* the consequent of his expression (16) as a premise to establish his first conclusion *via* a *reductio ad impossibile* argument. But we shall observe that if that first conclusion is in fact correct, then the antecedent of expression (16) must be false (or incorrect); and as a result, so must the consequent of expression (16) be. And of course, using a false premise in an argument renders the entire argument invalid.

We shall see this in greater detail in *(Lxxxii)* below.[65]

(Lxviii-B) Moreover, if Gödel's first conclusion after his expression (16) is taken into consideration, a contradiction is already implicit in **Proposition V**. Note that Gödel "proves" that his PROPOSITIONAL FORMULA **17 Gen r** is, as he puts it, "not κ-PROVABLE".[66] But such a conclusion would be contradictory to an implication of expression (3) of **Proposition V**, by which it can be proved that **17 Gen r** *is* κ-PROVABLE.

The argument is as follows.

For the purposes of this argument, let **P** denote a property — *any* property — belonging to *all* natural numbers, and let **p** denote a natural number — again, *any* natural number. Then, since by the above definition the property **P** belongs to *all* natural numbers, the natural number **p** *must* possess the property **P**.[67]

This fact can be written symbolically as **P(p)**. And because of the above definitions of **P** and **p**, the FORMULA **P(p)** *must* hold: *i.e.*, **P(p)** must be a theorem.

Now by expression (3), namely:

$$\mathbf{R(x_1 \ldots x_n) \rightarrow Bew\{Sb[r\ (u_1 \ldots u_n)|(Z(x_1) \ldots Z(x_n)]\}} \qquad (3)$$

… when **n=1,** the letter **R** is replaced by the letter **P**, and the letter **r** by the letter **p**, it should be the case that **P(p) → Bew{Sb[p u|(Z(p)]}**.

That is to say, by expression (3), if the natural number **p** were to possess the property **P**,[68] the PROPOSITIONAL FORMULA **Sb[p u|(Z(p)]** would have to be PROVABLE.

And since as we have seen, the natural number **p** *must* possess the property **P**, the PROPOSITIONAL FORMULA **Sb[p u|(Z(p)]** *must* be PROVABLE.

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[65] The argument given here requires one to have already read Gödel's Paper, so the first-time reader is advised to skip this part and return to it after having read **PART 2-B** of our critique, at which time it will make better sense.

[66] We are again jumping ahead a bit in our critique here, and the first-time reader would be again advised to read to the end of **PART 2-B** and then return to this argument, at which time it will make quite good sense.

[67] For the restricted purposes of this argument we shall take the bold letter '**P**' to denote, not the object-language **P**, but a property of natural numbers. We do this in order to bring our argument in line with Gödel's own symbolism.

[68] When a "relation" holds with respect to only one number, of course, it is more appropriately called a "property" than a "relation". (See in this regard also page *xv* of *Principia Mathematica*, Second Ed., Vol. I).

Now by expression (13), **Sb[p 19|(Z(p)] = 17 Gen r**. Thus if **17 Gen r** is not κ-PROVABLE, then neither is **Sb[p 19|(Z(p)]**.

However, it is to be noted now that the first symbol **p** in **Sb[p u|(Z(p)]** represents a CLASS-SIGN, and a CLASS-SIGN can have only *one* FREE VARIABLE; and as a consequence, in the two PROPOSITIONAL FORMULAE **Sb[p 19|(Z(p)]** and **Sb[p u|(Z(p)]**, the FREE VARIABLE **u** must be identical to the FREE VARIABLE **19**. Thus the FORMULA **Sb[p u|(Z(p)]** must be identical to the FORMULA **Sb[p 19|(Z(p)]**.

But if expression (3) holds, and there *is* a property **P** possessed by the natural number **p**, then **Bew{Sb[p u|(Z(p)]}** *must* hold — *i.e.*, **Sb[p u|(Z(p)]** must be PROVABLE; and by the identity of **Sb[p u|(Z(p)]** with **Sb[p 19|(Z(p)]**, the FORMULA **Sb[p 19|(Z(p)]** must also be PROVABLE.

Restricting oneself, then, to the class κ of ω-consistent FORMULAE, **Sb[p 19|(Z(p)]** *must* be κ-PROVABLE.

And of course, by the identity of **Sb[p 19|(Z(p)]** with **17 Gen r** given in Gödel's expression (13), the FORMULA **17 Gen r** must also be κ-PROVABLE … which however contradicts Gödel's first conclusion.

It is seen from the above argument that Gödel's conclusion, derived with the help of **Proposition V**, contradicts another conclusion derivable from **Proposition V**. This demonstrates that the method used to "prove" **Proposition V** must itself be inconsistent — *i.e.*, Gödel's methodology must be inconsistent.[69]

It is to be noted, however, that this inconsistency arises only as a result of accepting Gödel's definitions 1. to 46. as valid. In actual fact, as we pointed out in **PART 2-A**, they are not valid, because they are syncretic: they include terms from both the object-language and the metalanguage. In the FORMULA **Sb[p 19|(Z(p)]**, for instance, the symbol '**p**' immediately to the right of the combination of symbols '**Sb[**' belongs to the metalanguage, while the '**p**' in the combination of symbols '**Z(p)**' belongs to the object-language. As a result, the entire term '**Sb[p 19|(Z(p)]**' is actually quite meaningless. If metalanguage terms in it were written in a different font, as earlier, it would be written (for example) as ***Sb[p 19|(Z*(p)]**, which would immediately reveal the fact that the two **p**'s belong to different languages, and therefore, according to the use / mention rules for soundly establishing a proof in a metalanguage, may not be used in a single formula.

GÖDEL'S DEFINITION OF "ω-INCONSISTENCY"

(Lxix) Now Gödel's gives his definition of the condition termed by him "ω-consistency":

[69] Note that this only transpires *after* Gödel's having defined the term '**Bew**', using his method of "Gödelisation". Consequently the problem must lie in the process by which the term '**Bew**' is defined, *i.e.*, in the "Gödelisation".

> We now come to the object of our exercises: Let **c** be any class of FORMULAE.[70] We denote by **Flg(c)** (set of consequences of **c**) the smallest set of FORMULAE which contains all the FORMULAE of **c** and all AXIOMS, and which is closed with respect to the relation "IMMEDIATE CONSEQUENCE OF". **c** is termed ω-consistent, if there is no CLASS-SIGN **a** such that:
>
> **(n)[Sb(a v|Z(n)) ∈ Flg(c)] & [Neg(v Gen a)] ∈ Flg(c)**
>
> where **v** is the FREE VARIABLE of the CLASS-SIGN **a**.

By this definition Gödel attempts to differentiate between, on the one hand, *(a)* the *provability* of a FORMULA in a particular class **c** of FORMULAE, and on the other, *(b)* the *correctness* of that very FORMULA.

In other words, what he intends to do is to rely on the widely-held notion that if a proposition is provable, it *must* be correct; but on the other hand, if it is correct, it is not *necessarily* provable. (It *could* be so, of course, but is not *necessarily* so.) This is of course quite true in real life, for it is impossible to claim that we have the skills to prove *everything* that's true — if we did, we'd be omniscient, or at least pretty darn close!

But although this principle *does* apply in real life, it does not necessarily apply to all *formal* systems of logic. Indeed it has already been proven that it does *not* apply to the more simple symbolic logics, such as first order logic. These are *known* to be both consistent *and* complete: *i.e.*, in such logics, all true propositions — and *only* true propositions — are provable.

However, it is essential to Gödel's overall strategy for him to claim that this is not the case for any system of logic capable of formalising mathematics, or even simple arithmetic. Note that it is his contention, from his words in ***Section 1*** of his Paper, that:

> From the remark that **[R(q); q]**[71] asserts its own unprovability, it follows at once that **[R(q); q]** is correct, since **[R(q); q]** is certainly unprovable (because undecidable). So the proposition which is undecidable *in the system **PM*** [72] yet turns out to be decided by metamathematical considerations.

As we have seen earlier, to bolster his claim, Gödel has devised his method of "Gödelisation" or "Gödel-numbering" — namely, the assigning of natural numbers in one-to-one correspondence to every basic sign of the system **P**, as well as to every FORMULA and every PROOF — *i.e.,*

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[70] In Gödel's original Paper this class is represented by the Greek letter κ — most likely because in German, the word "class" is written as *Klasse* — but since using κ renders the document typographically highly intensive, we shall from this point onward adhere to Meltzer's convention of replacing the Greek letter κ by the Roman letter **c** (representing the English word "class".) Of course it goes almost without saying that Gödel intends to have the system **P** be an instance of the class **c** of ω-consistent FORMULAE. It is his intention, in fact, to have the class **c** represent *any* system of logic capable of formalising simple arithmetic.

[71] The formula that corresponds to **[R(q); q]** in ***Section 2*** of Gödel's Paper is **17 Gen r**.

[72] Gödel's emphasis.

to every *series* of basic signs — of the system **P** (or of any other system of logic capable of formalising simple arithmetic). Thereby he attempts to show that if a system of logic is capable of formulating simple arithmetic, then all its symbols, formulae and proofs can be "numbered", just as chapters in a book or houses on a street can be numbered; and once that is done, it is always possible to enunciate a proposition to the effect that "Formula No. so-and-so is unprovable".

Suppose for example — and for the sake of simplifying the argument — that "so-and-so" above represents the arbitrary number **32768**; if then formula No. **32768** turns out to be the very formula which asserts "Formula No. **32768** is unprovable", then that formula must assert its *own* unprovability, and therefore must be true even if it is false[73] — as Gödel indicates in his words quoted above.

However, as we observed in both **PART 1** as well as in **PART 2-A**, the "number" **32768** is not really a number in the mathematical sense — *i.e.*, it does not belong to the object-language **P**, namely formalised mathematics, in which Gödel tries to prove the existence of undecidable propositions — but is rather a *label* applied to that formula, and thus belongs to his metalanguage **G**. We see this more clearly when we use certain types of notation, as for example the North American way of writing "No. **32768**". This is quite often written in North America as "#**32768**", where the symbol '#' clearly indicates that what follows is not a number in the mathematical sense — one with or upon which mathematical operations can be meaningfully carried out. (For example, the expression "#**32768** ÷ #**10240**" would be meaningless, whereas the expression "32768 ÷ 10240" would have both a clear meaning as well as a correct mathematical answer, *viz.*, 3.2).

Russell and Whitehead also employ a similar system in "numbering" the formulae of the system **PM**: in *Principia Mathematica* all formulae are "numbered" with numbers that begin with an asterisk, for example thus: ∗**13.101** — thereby making it clear that the term ∗**13.101** in *Principia Mathematica* is not the same thing as the mathematical number 13.101.

If therefore we were to use the North American notation, then we find that formula #**32768** states "Formula #**32768** is unprovable", in which case it must be talking *about* formula #**32768**; and thus formula #**32768** would have to be in both the object-language and the metalanguage, which runs counter to the use / mention rules for soundly establishing a proof in a metalanguage.

The problem lies, as we have seen — and as we shall also see when discussing his expressions (8.1) to (16) in *(Lxxviii)* below *et seq.* (*q.v.*) — in the fact that Gödel requires, in order to "prove" his Theorem, that the Gödel-numbers assigned according to his method of "Gödelisation" be capable of being *used* in the FORMULAE of the system **P**, by substituting the Gödel-number of one of the FORMULAE — say, the FORMULA denoted by the Gödel-number **p** — *in* another FORMULA, say the FORMULA denoted by the Gödel-number **r**.

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[73] If true, of course, it *is* unprovable; while if false, it is *not* unprovable, *i.e.*, it is *provable*. And if provable, it must be true, for that's what "being provable" means.

But such substitution, as we showed in **PART 1** of out critique, runs absolutely contrary to the use / mention rules for soundly establishing a proof in a metalanguage. If as Gödel intends, a particular FORMULA is to be *proven* UNPROVABLE in **c** and yet *proven* correct, Gödel ought to:

(a) establish a consistent class of FORMULAE *different from* **c**[74] — let's call it the class of FORMULAE **g**[75], for "gödel" — under which it *can* be proved correct,

... and

(b) establish that the FORMULA which is UNPROVABLE-in-**c** is *provable-in-***g**.

But he does *not* establish such a consistent class of FORMULAE **g**, *different* from **c**. On the contrary, he tacitly claims that the Gödel-numbers of the FORMULAE of **c** *also* belong to **c**, so that a Gödel-number may be substituted *in* the class **c** for a FORMULA *of* the class **c**, and *vice versa*. (He does not say so explicitly, but as we shall demonstrate just a few paragraphs below, he does do so implicitly.)

However, this runs counter to the use / mention rules referred to earlier. Indeed the consequence of such a claim is, that there can *never* be a class **c** of FORMULAE that is even consistent — what to speak of ω-consistent!

For there must *always* be a Gödel-number for both: *(a)* on the one hand, every FORMULA expressible in the system **P**, *and (b)* on the other, for its NEGATION: since both of them — every FORMULA *and* every NEGATION of every FORMULA — must be composed of a series of basic signs of the system **P**, and thus each of them must correspond to *some* Gödel-number.

As a consequence, if — for example — both the FORMULAE **(∃x_1) x_1 B** {**(n)[Sb(a v**|Z**(n))**} and **(∃x_2) x_2 B {[Neg(v Gen a)]}** are expressible in terms of the basic signs of the system **P**,[76] there must always be a Gödel-number for each of them. Let us, for the sake of convenience, call them the Gödel-numbers **h** and **i** respectively.

Now if it be claimed that both **h** and **i**, being Gödel-numbers, belong to the system **P**, then by simply substituting the Gödel number **h** with the FORMULA **(∃x_1) x_1 B** {**(n)[Sb(a v|Z(n))**} and the Gödel-number **i** with the FORMULA **(∃x_2) x_2 B** {**[Neg(v Gen a)]**}, the system **P** would be rendered ω-inconsistent: for both these FORMULAE, taken together, signify that there is a proof in the system **P** for the FORMULA **(n)[Sb(a v|Z(n))** as well as for the FORMULA **Neg(v Gen a)**.

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[74] **c** corresponds, in Gödel's Theorem, to the object-language, the system **P**.

[75] **g** corresponds here to his metalanguage **G**.

[76] Note that it is one objective of Gödel's *soi-disant* "series of functions (and relations) 1-45" to try and show that every single one of them is expressible in terms of the basic signs of the system **P**. Thus if he is correct, all FORMULAE of the form **x B y** — examples of which are **(∃x_1) x_1 B** {**(n)[Sb(a v**|Z**(n))**} and **(∃x_2) x_2 B {[Neg(v Gen a)]}** — must also be expressible in terms of the basic signs of the system **P**. (If that were not so, of course, then his celebrated FORMULA **17 Gen r** would not belong *to* the system **P**, and thus could not be undecidable *in* the system **P** — contrary to Gödel's claim.)

On the other hand, if it be claimed that Gödel-numbers do *not* belong to the system **P**, then it would *not* be permissible *in* the system **P** to substitute a Gödel-number by any FORMULA *of* the system **P**, or *vice versa*.

In which case, however, it would not be possible to derive Gödel's "undecidable" PROPOSITIONAL FORMULA **17 Gen r**, for according to his expression (13) below (*q.v.*), the FORMULA **17 Gen r** can only be derived by substituting, in the FORMULA **Sb(p 19|Z(p))**, the Gödel-number **p** with the FORMULA **17 Gen q**.[77]

From the above, it follows that the establishment of a *consistent* class of FORMULAE **c** using the procedure of "Gödelisation" — *i.e.*, a procedure which does *not* distinguish between terms of object-language and those of the metalanguage, so that a Gödel-number *may* substitute or be substituted by the FORMULA to which it corresponds, and *vice versa* — is impossible.

On the other hand, it also follows that if in a class of FORMULAE **c**, a Gödel-number may *not* be substituted by the FORMULA belonging to the class of FORMULAE **c**, and *vice versa*, then although the class of FORMULAE **c** *would* be consistent and could even be ω-consistent, it would be quite impossible to derive Gödel's FORMULA **17 Gen r** *in* that class of FORMULAE **c**. That is to say, if a distinction between object-language terms and metalanguage terms *is* made, it is impossible to claim that the metalanguage formula **17 Gen r** belongs to the object-language **P** (which latter, of course, Gödel implicitly equates with the class of FORMULAE **c**.)

And if **17 Gen r** does *not* belong to the object-language **P**, then of course it cannot be undecidable *in* the object-language **P**. Even if it is undecidable, it must be undecidable in some other language — a language to which it *does* belong.

In an endeavour to express the above a bit more clearly, we request the reader to carefully consider Gödel's expressions (11) to (13) — repeated here below for the sake of ease of reference:

p = 17 Gen q (11)
(**p** is a CLASS-SIGN with the FREE VARIABLE **19**)

and

r = Sb(q 19|Z(p)) (12)
(**r** is a recursive CLASS-SIGN with the FREE VARIABLE **17**).

Then

Sb(p 19|Z(p)) (13)
= Sb ([17 Gen q] 19|Z(p))

[77] We are once again jumping ahead a bit in our critique here, and the first-time reader would again be advised, if he can't follow our reasoning here, to read to the end of **PART 2-B** and then return to this argument, at which time it will make quite good sense.

= 17 Gen Sb(q 19|Z(p))
= 17 Gen r

[because of (11) and (12)] …

Now if it be assumed that Gödel-numbers do *not* belong to the class **c** of FORMULAE, then if the FORMULA **17 Gen q** *does* belong to the class **c** of FORMULAE, the Gödel-number **p** may *not* be substituted in the class of FORMULAE **c** by the FORMULA **17 Gen q**: in which case the equality or identity

Sb(p 19|Z(p)) = Sb ([17 Gen q] 19|Z(p))

… of expression (13) cannot hold.

And likewise, if it be assumed that Gödel-numbers do *not* belong to the class **c** of FORMULAE, then the Gödel-number **r** may *not* substitute the FORMULA **Sb(q 19|Z(p))**: in which case the equality or identity

17 Gen Sb(q 19|Z(p)) = 17 Gen r

… of expression (13) cannot hold.

In either case, it would not be possible to derive the FORMULA **17 Gen r** in the class **c** of ω-consistent FORMULAE.

Whereas if it be assumed that Gödel-numbers *do* belong to the class **c** of FORMULAE, then the Gödel-number **p** *may* be substituted by the FORMULA **17 Gen q**, and the Gödel-number **r** *may* substitute the FORMULA **Sb(q 19|Z(p))** — in which case the FORMULA **17 Gen r** *can* be derived, as per expression (13), in the class **c** of FORMULAE.

But under this assumption, by the process of Gödelisation or Gödel-numbering, there must *also* exist Gödel numbers for the FORMULAE

$$(\exists x_1)\ x_1\ B_c\ \{[Neg(17\ Gen\ r)]\}$$

… and

$$(\exists x_2)\ x_2\ B_c\ \{(n)[Sb(r\ 17|Z(n))\}$$

— which we may designate here by the letters **j** and **k** respectively.

And since, under the above assumption, **j** and **k** *do* belong to the class **c** of FORMULAE, and as a consequence *may* be substituted *in* the class **c** by FORMULAE *of* the class **c** — just as the Gödel-number **p** may be substituted in the class **c** by the FORMULA **17 Gen q** — then by substituting the Gödel-number **j** by the FORMULA

$$(\exists x_1)\ x_1\ B_c\ \{[Neg(17\ Gen\ r)]\}$$

… and the Gödel-number **k** by the FORMULA

$$(\exists x_2)\ x_2\ B_c\ \{(n)[Sb(r\ 17|Z(n))\},$$

… the class **c** of FORMULAE would be rendered ω-inconsistent.

Thus either the class **c** of FORMULAE must be ω-inconsistent, or else, if the class **c** of FORMULAE is ω-consistent, then the FORMULA **17 Gen r** cannot be derived in it.

And as a consequence, if it be further assumed — as Gödel obviously, albeit tacitly, assumes — that the system **P** is ω-consistent, then the FORMULA **17 Gen r** cannot be derived in it.

Thus the FORMULA **17 Gen r** cannot belong to the system **P**. And as a consequence, neither can the FORMULA **Neg(17 Gen r)**.

So although it would indeed be quite correct, as Gödel claims, that neither **17 Gen r** nor **Neg(17 Gen r)** is **c**-PROVABLE, the reason for that would be, simply, that neither **17 Gen r** nor **Neg(17 Gen r)** can *belong* to an ω-consistent class **c** of FORMULAE such as the system **P**.

And consequently there would *not* be any undecidable FORMULA *in* an ω-consistent class **c** of FORMULAE such as the system **P**. Instead, any undecidable FORMULA, if it existed at all, would have to be *outside* every ω-consistent class **c** of FORMULAE.

(Lxx) Gödel now says:

> Every ω-consistent system is naturally also consistent. The converse, however, is not the case, as will be shown later.

We shall undertake an analysis of this statement later in *(Lxxxviii)* and *(Lxxxix)* — (*q.v.*) — when Gödel differentiates between an ω-consistent class **c** of formulae and a "merely" consistent class **c'** of (other) formulae.

GÖDEL'S "PROPOSITION VI"

(Lxxi) Gödel now says:

> The general result as to the existence of undecidable propositions reads:
>
> **Proposition VI**: To every ω-consistent recursive class **c** of FORMULAE there correspond recursive CLASS-SIGNS **r**, such that neither **v Gen r** nor **Neg (v Gen r)** belongs to **Flg(c)** (where **v** is the FREE VARIABLE of **r**).
>
> Proof: Let **c** be any given recursive ω-consistent class of FORMULAE. We define:
>
> $$Bw_c(x) \equiv (n)[n \leq l(x) \rightarrow Ax(n\ Gl\ x) \vee (n\ Gl\ x) \in c \vee (\exists p,q)\{0 < p,q < n\ \&\ Fl(n\ Gl\ x,\ p\ Gl\ x,\ q\ Gl\ x)\}]\ \&\ l(x) > 0 \quad (5)$$

(cf. the analogous concept 44)

Expression (5) is Gödel's definition of the concept "PROVABLE FORMULA belonging to the class **c** of ω-consistent recursive FORMULAE".

(Lxxii) Gödel's next expression is:

$\mathbf{x\ B_c\ y \equiv Bw_c(x)\ \&\ [l(x)]\ Gl\ x = y}$ (6)

Expression (6) is a similar definition of the concept "**x** is a PROOF of the FORMULA **y** belonging to the class **c** of ω-consistent recursive FORMULAE".

(Lxxiii) Gödel now says:

$\mathbf{Bew_c(x) \equiv (\exists y)\ y\ B_c\ x}$ (6.1)

(cf. the analogous concepts 45, 46)

The concept $\mathbf{y\ B_c\ x}$ which was defined in Gödel's expression (6), and the concept $\mathbf{Bew_c(x)}$, that is, "**x** is a PROVABLE FORMULA belonging to the class **c** of FORMULAE", are now shown to be equivalent.

The clear objective of Gödel's expressions (5) to (6.1) above and (8) below (*q.v.*) is the demonstration of the transformability of the symbols **Bw(x)** to **Bew(x)** to $\mathbf{Bew_c(x)}$ to $\mathbf{x\ B_c}$ and *vice versa*.

(Lxxiv) Gödel now says:

The following clearly hold:

$\mathbf{(x)[Bew_c(x) \sim x \in Flg(c)]}$ (7)[78]

$\mathbf{(x)[Bew(x) \rightarrow Bew_c(x)]}$ (8)

Expression (7) states that **x** is a **c**-PROVABLE FORMULA if and only if **x** belongs to the set of consequences of **c**, and *vice versa*. Expression (8) states that all PROVABLE FORMULAE — in Gödel's present argument, of course — belong to the class **c** of FORMULAE.

GÖDEL'S EXPRESSIONS (8.1), (9) AND (10)

(Lxxv) Gödel continues:

We now define the relation:

[78] In Gödel's notation, an expression of the form '***p* ~ *q***' means "either both ***p*** and ***q*** or both **~*p*** and **~*q***".

Q(x,y) ≡ ~{x B_c [Sb(y 19|Z(y))]}. (8.1)

Since **x B_c y** [according to (6), (5)] and **Sb(y 19|Z(y))** (according to definitions 17, 31) are recursive, so also is **Q(x,y)**. According to Proposition V and (8) there is therefore a RELATION-SIGN **q** (with the FREE VARIABLES **17**, **19**) such that

~{x B_c [Sb(y 19|Z(y))]} → Bew_c[Sb(q 17|Z(x) 19|Z(y))]. (9)

x B_c [Sb(y 19|Z(y))] → Bew_c[Neg Sb(q 17|Z(x) 19|Z(y))]. (10)

We shall analyse these lines very, *very* carefully hereunder.

(Lxxv-A) In the first place, Gödel's expression (8.1), namely:

Q(x,y) ≡ ~{x B_c [Sb(y 19|Z(y))]}. (8.1)

… should be considered in great detail, for it is the trickiest part of Gödel's "proof".

First of all we note that in this expression there is once again no separation of metalanguage symbols from object-language symbols. If it were written correctly, using, say, a different font to express metalanguage symbols, it would be written for example thus:

Q(x,y) ≡ ~{*x B_c* [*Sb*(*y 19*|Z(y))]}. (8.1)

From this it is clearly seen that expression (8.1) is syncretic, and as such, by the use / mention rules for establishing a metamathematical proof, it cannot be validly used for such purpose.

On the other hand, we shall see that if we accept Gödel's expression (8.1) at face value — *i.e.* as an expression in which metalanguage terms are *not* separated from object-language terms — contradictions inevitably arise.

In order to see this, we begin by noting that Gödel does not derive expression (8.1) from any previously-proven or defined expression or equivalence: it is simply a definition. The relation **Q(x,y)** is *defined* as being the relation that holds between **x** and **y** in **~{x B_c [Sb(y 19|Z(y))]}**.

We further note that Gödel writes no quantification of any kind — neither universal nor existential — in front of the FORMULA **~{x B_c [Sb(y 19|Z(y))]}**.

Let us, as an initial assumption, take it that this is exactly what Gödel intended: in other words, that he did *not* intend to tacitly imply that there is any quantification on the VARIABLE **x**.[79]

In that case, if translated into natural language, the above definition would be:

[79] We shall examine each of the following three possibilities: *(i)* that Gödel intended there to be no quantification of any kind on **x** in the definiendum of expression (8.1); *(ii)* that he intended a tacit (or implied) existential quantification on **x** in that FORMULA; and *(iii)* that he intended a tacit (or implied) universal quantification on **x** in that FORMULA. As we shall see, regardless of what he intended, it becomes impossible for him to prove his Theorem.

> "The relation **Q** is defined as the relation between two natural numbers **x** and **y** when **x** does *not* correspond, *via* the process of "Gödel-numbering" described earlier, to a PROOF in the class **c** of the FORMULA **Sb(y 19|Z(y))**, this being the PROPOSITIONAL FORMULA which results when, in the CLASS-SIGN which corresponds — again *via* the process of "Gödel-numbering" — to the natural number **y**, the FREE VARIABLE which corresponds to the natural number **19** is replaced by the NUMBER-SIGN for the natural number **y**."

This, however, is exactly the same definition that would accrue if there were an existential quantification on **x** in the FORMULA **~{x B_c [Sb(y 19|Z(y))]}**. (Obviously if either **x** or **y** did *not* exist, there could be no relation **Q** between them!)

As a result, even though Gödel has not actually *written* a quantification in front of the FORMULA **~{x B_c [Sb(y 19|Z(y))]}**, there is at least a tacit (or implied) existential quantification there, and the completely-written FORMULA should be **(∃x,y) ~{x B_c [Sb(y 19|Z(y))]}**.[80]

Now it is to be noted that it is not possible to derive **(∃x,y) ~{x B_c [Sb(y 19|Z(y))]}** from any previous FORMULA of Gödel's. His definition 46., which reads:

$$\mathbf{Bew(x) \equiv (\exists y)\ y\ B\ x}$$

… does not allow **(∃x,y) ~{x B_c [Sb(y 19|Z(y))]}** to be derived from it, for when both the definiens and the definiendum of definition 46. are prefixed by a negation sign, and the appropriate substitutions of symbols made, the result is *not*

$$\sim \mathbf{Bew[Sb(y\ 19|Z(y))] \equiv (\exists x,y) \sim\{x\ B_c\ [Sb(y\ 19|Z(y))]\}},$$

… but rather

$$\sim \mathbf{Bew[Sb(y\ 19|Z(y))] \equiv \sim[(\exists x)\ \{x\ B_c\ [Sb(y\ 19|Z(y))]\}]},$$

… which is the same as

$$\sim \mathbf{Bew[Sb(y\ 19|Z(y))] \equiv (x) \sim\{x\ B_c\ [Sb(y\ 19|Z(y))]\}}.$$

Since by definition a well-formed formula can *only* be one that is derived from another well-formed formula with the help of the logical connectives ~, ∨, →, **&**, and ≡,[81] it is therefore clear that the expression **(∃x,y) ~{x B_c [Sb(y 19|Z(y))]}** cannot be a well-formed formula.

And of course if **(∃x,y) ~{x B_c [Sb(y 19|Z(y))]}** is not a well-formed formula, neither can **(∃x) ~{x B_c [Sb(y 19|Z(y))]}** be a well-formed formula, for the latter is derived from the former.

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[80] In *(Lxxv-E)* we shall examine the result of assuming that there is absolutely no quantification of any kind, whether tacit, implied or otherwise, before **~{x B_c [Sb(y 19|Z(y))]}**.

[81] See for example <http://www.cs.odu.edu/~toida/nerzic/content/logic/pred_logic/construction/wff_intro.html> (among other Web sites), as well as any introductory text book on logic, for a definition of "well-formed formula".

We can also see that **(∃x) ~{x B_c [Sb(y 19|Z(y))]}** cannot be a well-formed formula by the following argument: when the negation-sign is *removed* from it, **(∃x) ~{x B_c [Sb(y 19|Z(y))]}** becomes **(x) x B_c [Sb(y 19|Z(y))]**, which when interpreted as to content states that *every* natural number corresponds, *via* the process of "Gödel-numbering", to a proof in **c** of **Sb(y 19|Z(y))** … which is absurd in light of the process of "Gödel-numbering" described in Gödel's Paper.

Thus the expression **(∃x) ~{x B_c [Sb(y 19|Z(y))]}**, being derived from an absurd 'FORMULA' by the addition of a '~' sign, must itself be absurd, and thus cannot possibly be well-formed.

Considering then the interpretation of expression (8.1) with an implied existential quantification on the **x**, so that **Q(x,y)** is regarded as being identical to **(∃x) ~{x B_c [Sb(y 19|Z(y))]}**, the question of whether a PROOF in **c** of the PROPOSITIONAL FORMULA **Sb(y 19|Z(y))** exists or does not exist is inapplicable — *i.e.*, such a PROOF can neither exist nor not exist: since by the above argument, expression (8.1) is not, and cannot be, a well-formed formula of either the system **P** (*i.e.*, the class **c** of ω-consistent FORMULAE) *or* the metalanguage **G**.

On the other hand, if it were to be hypothesised, merely for the sake of argument, that it *is* applicable, we shall see that when considered in conjunction with expression (3), a PROOF of the PROPOSITIONAL FORMULA **Sb(y 19|Z(y))** *must* exist in **c**. This can be seen from the following:[82]

Let **Y** represent any property belonging to *all* natural numbers; and let **y** be a natural number. Since **Y** is a property of *all* natural numbers and **y** is a natural number, the (one-place) relation **Y(y)** *must* hold. Then by expression (3), namely:

$$R(x_1 \dots x_n) \rightarrow Bew\{Sb[r\ (u_1 \dots u_n)|(Z(x_1) \dots Z(x_n)]\} \qquad (3)$$

… when **R** is the property **Y** and **n=1**,

$$Y(y) \rightarrow Bew[Sb(y\ u|Z(y))]$$

Thus since **Y(y)** *must* hold, so must **Bew[Sb(y u|Z(y))]**.

And, of course, when **c** is a class of ω-consistent FORMULAE,

$$Y(y) \rightarrow Bew_c[Sb(y\ u|Z(y))]$$

Thus **Sb(y u|Z(y))** *must* be PROVABLE in **c** — which of course means that a PROOF of the PROPOSITIONAL FORMULA **Sb(y u|Z(y))** *must* exist in **c**.

And since **y** corresponds to a CLASS-SIGN, there can be only one FREE VARIABLE in it; which means that if there is a FORMULA such as **Sb(y 19|Z(y))** in **c**, then the FREE VARIABLE **u** must be identical to the FREE VARIABLE **19**. Consequently there *must* be a PROOF in **c** of the PROPOSITIONAL FORMULA **Sb(y 19|Z(y))** in **c**.

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[82] This is essentially the same argument given in *(Lxviii-B)* above (*q.v.*) with **Y** substituted for **P** and **y** for **p**.

However, since by Gödel's expression (13), the PROPOSITIONAL FORMULA **17 Gen r** is identical to the PROPOSITIONAL FORMULA **Sb(p 19|Z(p))**, and since **Sb(y 19|Z(y))** is **Sb(p 19|Z(p))** with **y** substituted for **p**,[83] **17 Gen r** must be identical to **Sb(y 19|Z(y))** also. Thus if the PROPOSITIONAL FORMULA **Sb(y 19|Z(y))** is **c**-PROVABLE, so must **17 Gen r** be.

This conclusion, however, contradicts Gödel's first conclusion, *viz.,* that **17 Gen r** is *not* **c**-PROVABLE. Consequently, either Gödel's first conclusion must be in error, or our initial hypothesis, *viz.,* that the a PROOF in **c** of the PROPOSITIONAL FORMULA **Sb(y 19|Z(y))** either exists or does not exist, must be wrong.

And either way, Gödel cannot prove his Theorem.

(Lxxv-B) Now in order to circumvent the above objection, let us assume that the quantification on **x** is tacitly assumed by Gödel to be universal rather than existential; or in other words, that the definiens of expression (8.1) should be written more fully as **(x) ~{x B_c [Sb(y 19|Z(y))]}**.

This is also consistent with Gödel's definition 46., which (as we saw above) reads:

Bew(x) ≡ (∃y) y B x

Thus the negation of **Bew(x)**, namely **~Bew(x)**, would have to be defined as **~[(∃y) y B x]**, which is the same as **(y) ~(y B x)**.

However, this would *still* not allow Gödel to prove his Theorem, for the following reason:

It is to be noted that according to the procedure of "Gödel-numbering", each basic sign and each series of basic signs of the system **P** is assigned a unique "Gödel-number".

Now Gödel tacitly implies that the system **P** and the class **c** are identical, for otherwise, even if he were to prove that there is an undecidable FORMULA in the class **c**, there would not necessarily be one in the system **P**.

So the definiens of (8.1), namely **(x) ~{x B_c [Sb(y 19|Z(y))]}**, which when interpreted states categorically that there is no PROOF in the class **c** of the PROPOSITIONAL FORMULA **Sb(y 19|Z(y))**, must *ipso facto* also state that there is no PROOF in the system **P** of the PROPOSITIONAL FORMULA **Sb(y 19|Z(y))**.

And this in turn means that in the system **P**, there does *not* exist a series of basic signs which constitutes a PROOF of **Sb(y 19|Z(y))**.

So then: according to the process of "Gödel-numbering", how can a "Gödel-number" **x** — as in the expression **~{x B_c [Sb(y 19|Z(y))]}** — be assigned to a series of basic signs of the system **P** that, by its very definition, does *not* exist in the system **P**? The very notion of assigning a specific "Gödel-number" **x** to a *non*-existent series of basic signs is patently ludicrous.

[83] Note that by Gödel's words in natural language found between expressions (14) and (15) in his 1931 Paper, **p** may be substituted for **y**; and thus the reverse substitution must also be allowed.

(Lxxv-C) Moreover, if the quantification on **x** in **~{x B_c [Sb(y 19|Z(y))]}** is taken to be universal, then expression (8.1) *as a whole* becomes self-contradictory, in that *it simultaneously demands and denies the existence of a natural number* **x** in the class **c** of ω-consistent FORMULAE.

For if its definiendum, namely **Q(x,y)**, is to belong to **c**, then there *must* exist in **c** a natural number **x** such that **Q(x,y)** holds — since if the natural number **x** did *not* exist in **c**, neither could the binary relation **Q(x,y)** exist in **c**.

But if **Q(x,y)** *does* exist in **c**, then since expression (8.1) when **x** is universally quantified is:

Q(x,y) ≡ (x) ~{x B_c [Sb(y 19|Z(y))]},

… **(x) ~{x B_c [Sb(y 19|Z(y))]}** must also exist in **c**.

And yet **(x) ~{x B_c [Sb(y 19|Z(y))]}**, when interpreted as to content, states categorically that there *is* no **x** such that **x** is a PROOF in **c** of **Sb(y 19|Z(y))** — *i.e.*, there does *not* exist a PROOF in **c** of **Sb(y 19|Z(y))**, said PROOF corresponding, *via* the process of "Gödelisation", to the natural number **x**.

So **(x) ~{x B_c [Sb(y 19|Z(y))]}**, when interpreted as to content, states categorically that **x** does *not* exist (or at least does not exist in **c**). In other words, **(x) ~{x B_c [Sb(y 19|Z(y))]}**, when interpreted as to its content, *denies* that there exists in **c** any natural number **x**[84] such that **x** is a PROOF of the PROPOSITIONAL FORMULA **Sb(y 19|Z(y))** in the class **c** of ω-consistent FORMULAE.

And yet **Q(x,y)**, when interpreted as to *its* content, categorically *demands* the existence in **c** of the natural number **x**: or in other words, if **Q(x,y)** exists in **c**, the natural number **x** *must* exist in **c** as well.[85]

Both the above, taken together, render expression (8.1) as a whole self-contradictory, and therefore inconsistent.

To express the above argument (partially) symbolically, [86] let us denote a class of consistent FORMULAE by the symbol **c'**,[87] and let us initially assume that expression (8.1) belongs to it. Then:

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[84] We can express the notion that a particular number does not belong to a class **c** of FORMULAE by saying that the FORMULA of that number — *i.e.*, the FORMULA to which that number is identical — does not belong to **c**. (The NUMBER-SIGN for that particular number would nicely fit the bill.)

[85] At best, if the natural number **x** did *not* exist in **c**, the only relation **Q** between the natural numbers **x** and **y** that could exist in **c** is the relation that holds between one natural number that does exist and another that doesn't: in which case, however, expression (8.1) could still not belong to **c**, because the definiens of (8.1) contains both **x** *and* **y**, and thus the definiens of (8.1) — taken by itself — could not belong to **c** if **y** belongs to **c** but **x** doesn't.

[86] Although the argument given here is expressed in symbols, it is not an argument demonstrating a *formal* contradiction — *i.e.*, a contradiction arising from the mere *form* of the FORMULAE involved, without any reference to their interpretation — but rather a contradiction depending on the *meanings* of the FORMULAE. However, since we are discussing the metalanguage and not the object-language, and since the metalanguage must be *about* the object-lang-

(I) [Expression (8.1)] $\in$ **c'** $\rightarrow$ **Q(x,y)** $\in$ **c'** *by expression* (8.1)

(II) **Q(x,y)** $\in$ **c'** $\rightarrow$ **[(x) ~{x** $\mathbf{B_c}$ **[Sb(y 19|Z(y))]}]** $\in$ **c'** *by expression* (8.1)

(III) **(x) ~{x** $\mathbf{B_c}$ **[Sb(y 19|Z(y))]}** $\rightarrow$ **x** $\notin$ **c'** *by Gödel's definition of* **~{x** $\mathbf{B_c}$ **[Sb(y 19|Z(y))]}**

(IV) **x** $\notin$ **c'** $\rightarrow$ **Q(x,y)** $\notin$ **c'** *by* **Q(x,y)** *being a function of* **x**

(V) **[Q(x,y)** $\in$ **c'] & [Q(x,y)** $\notin$ **c']** *contradiction – by lines (I) and (IV) above*

Therefore our initial assumption, *viz.*, that expression (8.1) belongs to a class of consistent FORMULAE such as **c'**, must be wrong.

Thus expression (8.1) obviously *cannot* belong to a class of consistent FORMULAE such as the class **c'**. And if expression (8.1) does not belong to a class **c'** of consistent FORMULAE, it must be inconsistent.

And, of course, if it cannot belong to a class **c'** of consistent FORMULAE, it cannot belong to a class **c** of ω-consistent FORMULAE either: for as per Gödel's earlier words, every ω-consistent class of FORMULAE must be consistent, and consequently, every inconsistent class of FORMULAE must also be ω-inconsistent.

Moreover, no other FORMULA deriving from expression (8.1) can belong to a class of consistent formulae either. All such FORMULAE must also be inconsistent.

Thus if expression (8.1) were denoted by the letter **W**, since **W** is inconsistent, when both the definiens and the definiendum of **W** is prefixed by a negation sign, the resulting expression — *i.e.*, the complement of **W** — must be inconsistent too.

That is to say, if the expression which results when both the definiens and the definiendum of **W** is prefixed by a negation sign were denoted by **W'**, this **W'**, namely

~Q(x,y) ≡ x $\mathbf{B_c}$ **[Sb(y 19|Z(y))]** **(W')**

… must be inconsistent too — and thus cannot belong to a class of consistent formulae either.

Now coming to expressions (9) and (10), it is to be noticed that they are derived from **W** and **W'** respectively: that is to say, expression (9), which when written out more fully reads:

Q(x,y) ≡ (x) ~{x $\mathbf{B_c}$ **[Sb(y 19|Z(y))]}** $\rightarrow$ $\mathbf{Bew_c}$**[Sb(q 17|Z(x) 19|Z(y))]** (9)

… is derived from **W**, while expression (10), which when written out more fully reads:

uage, it is by definition impossible to have a metalanguage argument without reference to its interpretation! *Formally*, expression (8.1) is not well-formed, and so cannot even be self-contradictory, but only nonsensical.

[87] This is in accordance with Gödel's own notation in *(Lxxxix)* below (*q.v.*).

$$\sim Q(x,y) \equiv (\exists x)\ x\ B_c\ [Sb(y\ 19|Z(y))]\} \rightarrow Bew_c[Neg\ Sb(q\ 17|Z(x)\ 19|Z(y))] \qquad (10)$$

… is derived from **W'**.

But as we have seen, **W** is self-contradictory. Thus expression (9) cannot under any circumstances hold, because, being derived from an inconsistent expression, it too must be self-contradictory.

Indeed this is seen from a careful analysis of expression (9) itself, which when interpreted as to its content, reads:

> "If, in the class **c** of ω-consistent FORMULAE, there is no **x** such that **x** is a PROOF of the PROPOSITIONAL FORMULA **[Sb(y 19|Z(y))]**, then the PROPOSITIONAL FORMULA **[Sb(q 17|Z(x) 19|Z(y))]** is **c**-PROVABLE."

But if there is no **x** in the class **c** of ω-consistent FORMULAE, since **Z(x)** is a function of **x**, there can be no **Z(x)** either in the class **c** of ω-consistent FORMULAE; and thus the PROPOSITIONAL FORMULA

$$[Sb(q\ 17|Z(x)\ 19|Z(y))]$$

… which contains **Z(x)**, cannot exist in the class **c** of ω-consistent FORMULAE.

And of course, a FORMULA that does not even *exist* in the class **c** of ω-consistent FORMULAE cannot possibly be **c**-PROVABLE.

Expressing it symbolically, if **f(x)** denotes any function of **x**,

(I) $(x) \sim\{x\ B_c\ [Sb(y\ 19|Z(y))]\} \rightarrow (x)\ x \notin c$ — *by Gödel's definition* 46.

(II) $(x)\ x \notin c \rightarrow (x)\ [Z(x)] \notin c$ — *by* $(x)\ x \notin c \rightarrow (x)\ f(x) \notin c$

(III) $(x)\ [Z(x)] \notin c \rightarrow Sb(q\ 17|Z(x)\ 19|Z(y)) \notin c$
by $(x)\ x \notin c \rightarrow (x)\ f(x) \notin c$ *and by the definition of* $Sb(q\ 17|Z(x)\ 19|Z(y))$

(IV) $Sb(q\ 17|Z(x)\ 19|Z(y)) \notin c \rightarrow x'\ B\ [Sb(q\ 17|Z(x)\ 19|Z(y))]\} \notin c$ — *by Gödel's definition* 45.

(V) $x'\ B\ [Sb(q\ 17|Z(x)\ 19|Z(y))]\} \notin c \rightarrow \sim Bew_c[Sb(q\ 17|Z(x)\ 19|Z(y))]$
by the four lines above

(VI) $\sim\{x\ B_c\ [Sb(y\ 19|Z(y))]\} \rightarrow Bew_c[Sb(q\ 17|Z(x)\ 19|Z(y))]$ — *expression* (9)

(VII) $Bew_c[Sb(q\ 17|Z(x)\ 19|Z(y))]\ \&\ \sim Bew_c[Sb(q\ 17|Z(x)\ 19|Z(y))]$
contradiction – by lines (V) and (VI) above

Similarly, since expression (10) is derived from **W'**, and since **W'** has already been proven inconsistent (because the inconsistent expression **W** can be derived from it), expression (10) must be inconsistent too.

And since, as we shall see in *(Lxxxi)* below (*q.v.*), Gödel's expressions (15) and (16) are directly derived from his expressions (9) and (10), expressions (15) and (16) must also be inconsistent.

And most importantly, it is further to be noted that all these inconsistencies could never have arisen had the above-described syncretism *not* been introduced into the argument *via* the process of "Gödel-numbering" or "Gödelisation": and in particular, had **x** *not* been used in a single expression to denote *both* a natural number (*i.e.* an element of the object-language **P**) *and* a (non-existent) PROOF in **c** of a PROPOSITIONAL FORMULA (*i.e.*, an element of the metalanguage **G**). For it is to be borne in mind that expression (8.1) is of the form

$$\mathbf{F(x) \equiv \sim f(x)}$$

… where **F** and **f** denote different functions of **x**; and such an expression is not inconsistent at all, *provided* of course that the symbol **x** represents *exactly* the same concept in *both* the definiens *and* the definiendum of said expression — as would be the case when both belong to the object-language.[88]

When however a *single* symbol is made to represent two *different* concepts, one concept belonging to the object-language and the other to the metalanguage — *i.e.*, when syncretism creeps into the expression — inconsistencies are inevitable. In fact, this is precisely the reason the above-noted use / mention rules for soundly establishing a metamathematical proof have been enunciated in the first place.

(Lxxv-D) It is also to be noted that in expression (8.1), **Q, x**, **y** and **19** are chosen entirely arbitrarily. They are not derived from any previously-proved formulae. Thus **Q** represents *any* relation, **y** represents *any* CLASS-SIGN, and **19** *any* FREE VARIABLE. This has serious consequences for the implications of Gödel's conclusions, as we shall see in *(Lxxxvi)* below.

(Lxxv-E) Now let us tackle the question of whether the definition in expression (8.1) *exactly* as it is written by Gödel — *i.e.*, without any quantification whatsoever, and assuming that there is no quantification implied or otherwise on **x** — is a valid definition in logic. This is partly a philosophical question, since there is no universally accepted definition of "definition" in logic.

However, if it be assumed for the sake of argument that the definition of "definition" is the same as the definition of "identity" — *i.e.*, if a formula is defined as being the same as another, then the one is identical to the other — then by ∗**13.101** of *Principia Mathematica*, namely $\vdash\!: x = y \,.\, \psi x \supset \psi y$, which is explained by its authors as "*I.e.* if *x* and *y* are identical, any property

[88] For example, if **f(x)** is $\mathbf{x^2+1}$ and **F(x)** is $\mathbf{x^3+2}$, the expression $\mathbf{F(x) \equiv \sim f(x)}$ — or $\mathbf{x^3+2 \equiv \sim(x^2+1)}$ holds for all **x** when **x** represents a natural number (and *only* a natural number).

of *x* is a property of *y*", if **Q(x,y)** is to be defined as being identical to **~{x B_c [Sb(y 19|Z(y))]}**, then every property of **Q(x,y)** must be identical to every property of **~{x B_c [Sb(y 19|Z(y))]}**.

Now although this *is* possible if expression (8.1) is taken in *isolation*, it is not then possible to claim — as Gödel does, in an attempt to derive his expressions (9) and (10) from (8.1) — that **Q(x,y)** is an example of the antecedent of expression (3), namely **R(x_1 … x_n)**. In expression (3), **R(x_1 … x_n)** represents a relation **R** that holds between — as Gödel writes — an **n**-tuple of *numbers* **(x_1 … x_n)**. That is to say, the expression **R(x_1 … x_n)** belongs to the object-language **P**. But **~{x B_c [Sb(y 19|Z(y))]}** belongs to the metalanguage **G**: in it, **x** and **y** represent natural numbers which *correspond*, *via* a specific process (*viz.*, the process of "Gödel-numbering" or "Gödelisation"), to specific strings of symbols of the system **P**.

However, if **~{x B_c [Sb(y 19|Z(y))]}** and **Q(x,y)** are to be identical to one another, then they ought to belong to the same language: for by the definition of "identity", every property of the **Q(x,y)** must be a property of **~{x B_c [Sb(y 19|Z(y))]}**.

Thus **Q(x,y)** *cannot* be considered to be an instance of **R(x_1 … x_n)**: *i.e.*, **Q(x,y)** cannot be considered identical to the antecedent of expression (3), *viz.*, **R(x_1 … x_n)**, when **R=Q**, **n=2**, **x_1=x** and **x_2=y**. After all, there can be no identity between one term belonging to the metalanguage and another belonging to the object-language: since there is a property possessed by the former, *viz.,* that of belonging to the metalanguage, which the latter does not possess … and *vice versa*, there is a property possessed by the latter, *viz.,* that of belonging to the object-language, which the former does not possess.

And, of course, if there is no identity between **Q(x,y)** and **~{x B_c [Sb(y 19|Z(y))]}**, the definition in expression (8.1) as it is written by Gödel — *i.e.*, without any quantification whatsoever, and assuming that there is no quantification implied or otherwise on **x** — cannot possibly hold.

(Lxxvi) Now coming again to Gödel's expressions (9) and (10), *both* the antecedents and/or *both* consequents of (9) and (10) — like the antecedents and consequents of expressions (3) and (4) of **Proposition V** — cannot be held *simultaneously*.

Either one *or* the other antecedent can hold, but not *both simultaneously:* for if they did, the FORMULA

~{x B_c [Sb(y 19|Z(y))]}

… and the FORMULA

x B_c [Sb(y 19|Z(y))]

… which contradict one another, would have to hold simultaneously: rendering inconsistent the system of logic under which expressions (9) and (10) are derived.

And likewise, *either* the consequent of expression (9) *or* the consequent of expression (10) can hold, but not *both simultaneously:* for if they did, the FORMULA

Bew_c[Sb(q 17|Z(x) 19|Z(y))]

… and the FORMULA

$$\mathbf{Bew_c[Neg\ Sb(q\ 17|Z(x)\ 19|Z(y))]}$$

… would also have to hold simultaneously — *i.e.*, a FORMULA and its NEGATION would be **c**-PROVABLE simultaneously — which too would render inconsistent the system of logic under which expressions (9) and (10) are derived.

And since, as we shall see in *(Lxxxi)* below (*q.v.*), Gödel's expressions (15) and (16) are directly derived from his expressions (9) and (10) by substitution of **p** for **y**, neither can *both* the antecedents of expressions (15) and (16) be held simultaneously. Similarly, *both* their consequents cannot be held simultaneously either.

This must be borne in mind in the arguments that follow, for it will be found that Gödel attempts to use *both* the consequents of expressions (15) and (16) in a *single* argument, which as we see from the foregoing is obviously not permissible within the bounds of bivalent logic.

(Lxxvii) And in expressions (9) and (10) we find once again Gödel's principal mistake — the confusion of the metalanguage and object-language levels — as follows:

If Gödel's words

> … there is … a RELATION-SIGN **q** (with the FREE VARIABLES **17**, **19**)

… are to be taken as valid, then it is obvious that in his metalanguage **G**, the symbols **17** and **19** do not represent natural numbers, but FREE VARIABLES.

But *in the system **P***, the symbols **17** and **19** represent natural numbers, and do *not* represent FREE VARIABLES.

As a result, *neither* of Gödel's expressions

$$\mathbf{\sim\{x\ B_c\ [Sb(y\ 19|Z(y))]\} \rightarrow Bew_c[Sb(q\ 17|Z(x)\ 19|Z(y))]}. \qquad (9)$$

… and

$$\mathbf{x\ B_c\ [Sb(y\ 19|Z(y))] \rightarrow Bew_c[Neg\ Sb(q\ 17|Z(x)\ 19|Z(y))]}. \qquad (10)$$

… can be considered definable *in the system **P***.

And of course if they are not definable *in* the system **P**, Gödel's entire argument from this point onwards cannot apply *to* the system **P**, rendering it impossible for him to prove his Theorem *for* the system **P** — as is his intention according to his words at the beginning of Part 2 of his Paper, *viz.*,

> We … begin by giving an exact description of the formal system **P**, for which we seek to demonstrate the existence of undecidable propositions.

Now in rebuttal it may be argued that what Gödel *intends* to say is:

> … there is … a RELATION-SIGN **q** (with FREE VARIABLES corresponding to the natural numbers[89] **17**, **19**)

… rather than

> … there is … a RELATION-SIGN **q** (with the FREE VARIABLES **17**, **19**).

That is to say, Gödel uses the natural number **19** which, according to his scheme of "Gödelisation" or assigning natural numbers in one-to-one correspondence to basic signs and series of basic signs of the system **P**, *corresponds to* a VARIABLE in the FORMULA which corresponds *via* "Gödelisation" to the natural number represented by the symbol **y**. Such a VARIABLE would, in the system **P**, be ordinarily denoted by a letter of the alphabet such as **v** or **u**, with or without a subscript (such as $\mathbf{v_1}$ or $\mathbf{u_1}$).

And similarly, it may be argued that Gödel uses the natural number **17** which likewise *corresponds*, again *via* "Gödelisation", *to* a VARIABLE in the FORMULA which corresponds to the natural number represented by the symbol **q —** and which VARIABLE would, in the system **P**, ordinarily be denoted by another letter of the alphabet, or by the same letter with a different subscript.

If that is so, however, then we draw our readers' attention to the fact that the natural numbers **y** and **q**, as well as the NUMBER-SIGNS **Z(x)** and **Z(y)**, must *themselves* correspond to (other) natural numbers according to that self-same system of "Gödelisation" or "Gödel-numbering".

Suppose that **y** corresponds to the Gödel-number **g** and **Z(y)** corresponds to the Gödel-number **g'**; then if by the above reasoning it is permissible for Gödel to replace the letter **v** or **u** — which ordinarily represents a VARIABLE in the system **P** — for a symbol such as **19** which is defined as a natural number *corresponding to* the letter **v** (or **u** as the case may be) according to his system of "Gödel-numbering", it should also be permissible for us, according to that self-same system of "Gödel-numbering", to substitute the determinate natural number — the "Gödel-number" — to which **y** corresponds by the number **g**, and the determinate "Gödel-number" to which **Z(y)** corresponds by the number **g'**.[90] Then the expression

[Sb(y 19|Z(y))]

… could be written

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[89] We use underlining to indicate the words Gödel might be thought of as having intended to say, but didn't.

[90] The question essentially is whether it is possible for a rule to exist which permits the substitution of a term in the object-language by the Gödel-number — the metalanguage term — which corresponds to it, without the resulting formulae becoming meaningless. As we shall see, if such a rule exists, it is inevitable that the resulting formulae become meaningless. Yet without such a rule Gödel cannot even *attempt* to prove his Theorem: which shows that Gödel must rely on meaningless formulae — *i.e.*, those that are not well-formed — to try to prove his Theorem.

[Sb(g 19|g')]

Now, of course, the terms **g**, **19** and **g'** do represent three determinate natural numbers, and none of them represent FORMULAE, VARIABLES or NUMBER-SIGNS. However, the expression

[Sb(g 19|g')]

… is *still* not definable in the system **P**, for when it is "interpreted as to content" (as Gödel puts it), it signifies, in natural language,

> that formula which is derived when the determinate natural number **g'** is substituted for the natural number **19** in the determinate natural number **g.**

… and which in turn is utterly nonsensical in *any* interpretation of the system **P** in which "natural numbers" can be defined. (After all, there *is* no such thing as a "natural number **19** in the determinate natural number **g**"; and in any case, when one natural number is substituted for another, the resulting expression is not a FORMULA but another natural number.)[91]

As a result, of course, Gödel's expressions

~{x B_c [Sb(y 19|Z(y))]} → Bew_c[Sb(q 17|Z(x) 19|Z(y))]. (9)

… and

x B_c [Sb(y 19|Z(y))] → Bew_c[Neg Sb(q 17|Z(x) 19|Z(y))]. (10)

— which contain the expression **[Sb(y 19|Z(y))]** — are *still* not definable in the system **P**.

But this creates a serious problem (*i.e.*, for Gödel, of course): for his expressions

17 Gen r

… and

Neg(17 Gen r)

---

[91] It actually gets much worse: for if the numbers **g**, **19** and **g'** are taken to be natural numbers definable in accordance with formulae of the system **P**, then according to the system of "Gödel-numbering" there must correspond *yet* other natural numbers — call them **g''**, **g'''** and **g''''** — to them! In which case, the expression **[Sb(g 19|g')]** can also be written as **[Sb(g'' g'''|g'''')]**. And by that same reasoning, if the numbers **g''**, **g'''** and **g''''** are taken to be natural numbers definable in accordance with formulae of the system P, then according to the system of "Gödel-numbering" there must correspond yet other natural numbers — call them **g'''''**, **g''''''** and **g'''''''** — to them too … and so on *ad infinitum*. But since there is absolutely no guarantee that **g** is equal to **g''** or **g''** is equal to **g'''''**, *etc.,* there is no guarantee either that the FORMULA **[Sb(g'' g'''|g'''')]** is equivalent to the FORMULA **[Sb(g 19|g')]**, etc. — resulting in the blatant contradiction that *both* **{[Sb(g 19|g')] ≡ [Sb(g'' g'''|g'''')]}** *and* **~{[Sb(g 19|g')] ≡ [Sb(g'' g'''|g'''')]}** might well be correct (and so on *ad infinitum*).

— which according to his argument are supposed to be "not **c**-PROVABLE", but which are also derived from his expressions (9) and (10) above, also remain *not* definable in the system **P**.

And if they are not definable in the system **P**, Gödel's entire argument from this point onwards *still* cannot apply to the system **P**, rendering it impossible for him to prove his Theorem *for* the system **P** — as is his intention.

GÖDEL'S EXPRESSIONS (11) TO (16)

(Lxxviii) Gödel's next step is:

We put

p = 17 Gen q (11)
(**p** is a CLASS-SIGN with the FREE VARIABLE **19**)

and

r = Sb(q 19|Z(p)) (12)
(**r** is a recursive CLASS-SIGN with the FREE VARIABLE **17**).

Note that these expressions (11) and (12) are also definitions in Gödel's argument: that is to say, they are not derived from precedent derivation lines, neither are they axioms, nor are they theorems. Together with expression (8.1) they are Gödel's definitions. And as such, they are mere *premises*.

Thus also, both **p** and **r** are Gödel-numbers representing *arbitrarily* chosen CLASS SIGNS, and **17** — along with **19** — are Gödel-numbers representing arbitrarily chosen FREE VARIABLES.

As a consequence, **r** could represent *any* CLASS-SIGN in the class **c** of ω-consistent FORMULAE, and **17** could represent *any* FREE VARIABLE in the class **c** of ω-consistent FORMULAE. This, as we shall see later, has serious and adverse consequences for Gödel's conclusion.

Now it should in addition be obvious that **p** and **r** cannot be defined as Gödel intends them to be defined: namely, as Gödel-numbers representing CLASS-SIGNS.

For as we saw, $\mathbf{(x) \sim\{x\ B_c\ [Sb(y\ 19|Z(y))]\} \rightarrow x \notin c}$ — *i.e.*, the definiens of expression (8.1),[92] when written with Gödel's definition 46. and expression (6.1) in mind, implies that **x** cannot belong to **c**. And as a result, the definiendum of (8.1), *viz.,* the two-place relation **Q(x,y)**, also cannot belong to **c**. At best, the relation **Q** can only hold in **c** with respect to a *single* natural number **y**. (At worst, of course — *i.e.*, if it is insisted that **Q** is a *two*-place relation — then **Q**

[92] We use an implied universal quantification on **x** because, as we have seen, if the quantification is assumed to be existential, the definiens of expression (8.1) is no longer well-formed, while if it is assumed that there is *no* quantification of any kind in $\mathbf{\sim\{x\ B_c\ [Sb(y\ 19|Z(y))]\}}$, then expression (8.1) cannot possibly hold. (See *(Lxxv-A)* and *(Lxxv-E)* above.)

cannot exist at all in **c**, in which case neither can the RELATION-SIGN **q** exist in **c** … and in that case, Gödel's "proof" falls flat on its face.)

Now it may be argued that there is another interpretation of **Bew_c[Sb(q 17|Z(x) 19|Z(y))]**, based on Gödel's footnote No. 36, which reads:

[36] Where **v** is not a VARIABLE or **x** is not a FORMULA, then **Sb(x v|y) = x.**

(This footnote does not say what happens when **y** does not belong at all to the class of formulae to which **Sb(x v|y)** belongs; but we shall let that pass for now.)

In that case, **Bew_c[Sb(q 17|Z(x) 19|Z(y))] = Bew_c[Sb(q 19|Z(y))]** — *i.e.*, by expression (3) of **Proposition V**,

Q(y) → Bew_c[Sb(q 19|Z(y))].

But then — and contrary to Gödel's assertion in *(Lxxvi)* above — **q** must be, not a *two*-place RELATION-SIGN, but a *one*-place RELATION SIGN: *i.e.*, a CLASS-SIGN. A two-place RELATION-SIGN must contain — by its very definition — *two* FREE VARIABLES, not just one. And if **x** does not belong to **c**, neither can **Z(x)** … and thus neither can **Sb(q 17|Z(x) 19|Z(y))**.

And, by **Q(y) → Bew_c[Sb(q 19|Z(y))]**, that one FREE VARIABLE must be represented by the Gödel-number **19**, not by the Gödel-number **17**.

Thus a GENERALISATION of the CLASS-SIGN **q** cannot be carried out with the FREE VARIABLE **17** at all! In other words, Gödel's PROPOSITIONAL FORMULA **17 Gen q** — as given in his expression (11) — cannot hold at all, because there is no FREE VARIABLE **17** in **q**.

And even if it be argued — stretching the argument almost to breaking point — that *any* FREE VARIABLE would suffice to carry out the GENERALISATION of a CLASS-SIGN, the resulting FORMULA that emerges *after* such a GENERALISATION would be a PROPOSITIONAL FORMULA, not a CLASS-SIGN. And by Gödel's definition of PROPOSITIONAL FORMULA in *(xv)* above (*q.v.*), it cannot contain *any* FREE VARIABLE.

Thus — and utterly contrary to Gödel's words just below his expression (11) — **17 Gen q** could not possibly be equal to a CLASS-SIGN **p** … whether "**p** is a CLASS-SIGN with the FREE VARIABLE **19**" or not! For instead of representing a CLASS-SIGN, **p** would have to represent a PROPOSITIONAL FORMULA — *i.e.*, a FORMULA which, by Gödel's own definition in *(xv)* above (*q.v.*), contains *no* FREE VARIABLE at all.

Also, if **p** is to represent a PROPOSITIONAL FORMULA, **r** cannot be defined as equal (or identical) to **Sb(q 19|Z(p))** — as Gödel does in his expression (12) — because **p** does not represent a natural number at all, but a PROPOSITIONAL FORMULA — and it is meaningless for a PROPOSITIONAL FORMULA to have a NUMBER-SIGN.

And even if it be argued that the '**p**' in '**Z(p)**' does not represent the PROPOSITIONAL FORMULA **p**, but rather the natural number to which the PROPOSITIONAL FORMULA **p** *corresponds* by the process of "Gödelisation", it would still not detract from the fact that **Sb(q 19|Z(p))** would be a PROPOSITIONAL FORMULA itself — because as we saw above, **q** must be, not a *two*-place RELATION-SIGN, but a *one*-place RELATION SIGN: *i.e.*, a CLASS-SIGN; and thus **r**, if by expression (12) it is to be equal to the PROPOSITIONAL FORMULA **Sb(q 19|Z(p))**, would also be a PROPOSITIONAL FORMULA and not a CLASS-SIGN at all — directly contradicting Gödel's words just below expression (12).

(Lxxix-A) Gödel continues:

Then

Sb(p 19|Z(p)) (13)
= Sb ([17 Gen q] 19|Z(p))
= 17 Gen Sb(q 19|Z(p))
= 17 Gen r

[because of (11) and (12)] …

But if, as we have shown in *(Lxxviii)* above (*q.v.*), **r** is itself a mere PROPOSITIONAL FORMULA, by Gödel's own definitions in *(xv)* above (*q.v.*), it *cannot* have any FREE VARIABLE in it.

And if so, the question of any GENERALISATION of **r** by means of a FREE VARIABLE **17** — as Gödel implies in his expression (13) — cannot arise at all.

Or in other words, there cannot *be* any expression **17 Gen r** at all!

(Lxxix-B) Even if, by the implications of Gödel's footnote No. 18a, which says:

[18a] … **x ∀ (a)** is also a formula if **x** does not occur, or does not occur free, in **a**. In that case, **x ∀ (a)** naturally means the same as **a**.

… and even if it be further argued that **17 Gen r** is a "Gödelisation" of **17 ∀ r**, which in turn is an instance of **x ∀ (a)** with **x = 17** and **(a) = r**, the only thing such an argument can result in is the equivalence

17 Gen r ≡ r

…and even if it be argued in addition, that by the implications of Gödel's footnote No. 36 (repeated here below for the sake of ease of reference):

[36] Where **v** is not a variable or **x** is not a formula, then **Sb(x v|y) = x**.

… the only thing such an argument can result in is the equivalence

Sb(r 17|Z(x)) ≡ r

But in that case, expression (15) discussed in *(Lxxxi)* below, namely

$$\sim[x\ B_c\ (17\ Gen\ r)] \rightarrow Bew_c[Sb(r\ 17|Z(x))] \qquad (15)$$

… ends up reading

$$\sim[x\ B_c\ (r)] \rightarrow Bew_c(r) \qquad (15)$$

… or, what is the same thing,

$$\sim[Bew_c(r)] \rightarrow Bew_c(r) \qquad (15)$$

This, being interpreted as to content, states that if **r** is *not* **c**-PROVABLE, then **r** *is* **c**-PROVABLE — and therefore the proposition "**r** is **c**-provable" must be true (or correct).[93] That is to say, **r** must *be* **c**-PROVABLE.

And this means that if, as noted above, **17 Gen r ≡ r**, then **17 Gen r** must *also* be **c**-PROVABLE … contrary to Gödel's claim.

(Lxxx) Gödel continues:

> … and furthermore:
>
> $$Sb(q\ 17|Z(x)\ 19|Z(p)) = Sb(r\ 17|Z(x)) \qquad (14)$$
>
> [according to (12)].

As we saw in *(Lxxv-E)* above (*q.v.*), even if the syncretism inherent in Gödel's expressions (8.1) to (16) is disregarded, since by the definiens of expression (8.1), **x** does not belong to **c**, the two-place relation **Q(x,y)** also cannot belong to **c**. Consequently, the RELATION-SIGN **q**, even if it exists in **c** at all, is at best a *one*-place RELATION-SIGN — *i.e.*, a CLASS-SIGN — and can *never* be a *two*-place RELATION-SIGN. Consequently, no PROPOSITIONAL FORMULA **Sb(q 17|Z(x) 19|Z(p))** can belong to **c**, for it derives from a *two*-place RELATION-SIGN, and not from a CLASS-SIGN at all.

As a result, since **x** does not belong to **c**, and consequently **Z(x)** also does not belong to **c**, **Sb(q 17|Z(x) 19|Z(p))** must be taken to be the same as **Sb(q 19|Z(p))**.

Now **Sb(q 19|Z(p))**, by expression (12), is defined by Gödel to be equal to **r**. And since **q** must be a CLASS-SIGN and not a two-place RELATION-SIGN, **Sb(q 19|Z(p))** can only be a PROPOSITIONAL FORMULA. Consequently **r** also can only be a PROPOSITIONAL FORMULA, and not a CLASS-SIGN.

Thus there can never be any **Sb(r 17|Z(x))** in **c** either. At best, since **Z(x)** does not belong to **c**, **Sb(r 17|Z(x)) ≡ r**.

---

[93] We apply ∗2.18 of *Principia Mathematica*, which is — according to its authors — "the complement of the principle of the *reductio ad absurdum*." They add, in clarification: "It states that a proposition which follows from the hypothesis of its own falsehood is true."

But in that case, what expression (14) ends up saying is merely

$$\mathbf{r \equiv r}$$

… or, as Archie Bunker might have said, "*Whoop-ti-doo*!"

(Lxxxi) Gödel continues:

> If now in (9) and (10) we substitute **p** for **y**, we find, in virtue of (13) and (14):
>
> $$\mathbf{\sim[x\ B_c\ (17\ Gen\ r)] \rightarrow Bew_c[Sb(r\ 17|Z(x))]} \qquad (15)$$
>
> $$\mathbf{x\ B_c\ (17\ Gen\ r) \rightarrow Bew_c[Neg\ Sb(r\ 17|Z(x))]} \qquad (16)$$

But as we showed in *(Lxxix)* above (*q.v.*), neither **17 Gen r** nor **Sb(r 17|Z(x))** can exist in **c**. Or at best, when there is no **x** in **c**, both

$$\mathbf{17\ Gen\ r \equiv r}$$

… and

$$\mathbf{Sb(r\ 17|Z(x)) \equiv r.}$$

But in that case, expression (15) ends up reading

$$\mathbf{\sim[x\ B_c\ (r)] \rightarrow Bew_c(r)} \qquad (15)$$

… or, what is the same thing,

$$\mathbf{\sim[Bew_c(r)] \rightarrow Bew_c(r)} \qquad (15)$$

That is to say — and as we saw earlier too — **r** becomes **c**-PROVABLE.

And this means that if, as noted above, **17 Gen r ≡ r**, then **17 Gen r** must *also* be **c**-PROVABLE … contrary to Gödel's claim.

Besides, when expression (15) reads $\mathbf{\sim[x\ B_c\ (r)] \rightarrow Bew_c(r)}$, expression (16) becomes

$$\mathbf{x\ B_c\ (r) \rightarrow Bew_c(Neg\ r)} \qquad (16)$$

… or, what is the same thing,

$$\mathbf{Bew_c(r) \rightarrow Bew_c(Neg\ r)} \qquad (16)$$

… that is to say, a FORMULA and its NEGATION become provable in **c** at the same time, thus rendering the class **c** of FORMULAE inconsistent (and, *a fortiori*, ω-inconsistent) — contrary to Gödel's earlier assumption.

Even if it be argued hypothetically that when there is no **x** in **c**, **17 Gen r** and **Sb(r 17|Z(x))** are *not* the equivalent of **r**, expression (15) implies a contradiction, as can be seen below:

We first note that Gödel's expressions (15) and (16) are obtained interchanging the antecedents of expressions (9) and (10) for (13), and the consequents of (9) and (10) for the second term of (14). That is to say, in the ultimate analysis, expressions (15) and (16) are derived from expressions (9) and (10).

But since, as we have seen above in *(Lxxv)*, expressions (9) and (10) are inconsistent and cannot belong to a class **c** of ω-consistent FORMULAE, so also expressions (15) and (16) are inconsistent and cannot belong to a class **c** of ω-consistent FORMULAE, since they are derived from expressions (9) and (10).

And this, moreover, can be seen from a careful analysis of expression (15) itself, which when interpreted as to its content reads:

> "If, in the class **c** of ω-consistent FORMULAE, there exists no **x** such that **x** represents a PROOF of the PROPOSITIONAL FORMULA **17 Gen r**, then the PROPOSITIONAL FORMULA **Sb(r 17|Z(x))** is **c**-PROVABLE."

But if there exists no **x** in the class **c** of ω-consistent FORMULAE, since **Z(x)** is a function of **x**, there can exist no **Z(x)** either in the class **c** of ω-consistent FORMULAE; and thus the PROPOSITIONAL FORMULA

Sb(r 17|Z(x))

… in which the non-existent **Z(x)** is mentioned, also cannot exist in the class **c** of ω-consistent FORMULAE.

And of course, a FORMULA that does not even *exist* in the class **c** of ω-consistent FORMULAE cannot be PROVABLE in it — *i.e.*, **Sb(r 17|Z(x))** cannot be **c**-PROVABLE.

Expressing it symbolically, if **f(x)** denotes a function of **x**, if there is no **x** in the class **c** of ω-consistent FORMULAE, there can be no **f(x)** in the class **c** of ω-consistent FORMULAE either.

That is to say, $x \notin c \rightarrow f(x) \notin c$.

And then:

(I) $\sim\{x\ B_c\ (17\ Gen\ r)\} \rightarrow x \notin c$ *by Gödel's definition of* $\sim\{x\ B_c\ (17\ Gen\ r)\}$

(II) $x \notin c \rightarrow Z(x) \notin c$ *by* $x \notin c \rightarrow f(x) \notin c$

(III) $Z(x) \notin c \rightarrow [Sb(r\ 17|Z(x))] \notin c$ *by Gödel's definition of* $[Sb(y\ v|x)]$

(IV) $[Sb(r\ 17|Z(x))]\} \notin c \rightarrow \{x'\ B_c\ [Sb(r\ 17|Z(x))]\} \notin c$ *by Gödel's definition* 45.

(V) **{x' B_c [Sb(r 17|Z(x))]} ∉ c → ~Bew_c[Sb(r 17|Z(x))]** *by Gödel's definition* 46.

(VI) **~{x B_c (17 Gen r)} → ~Bew_c[Sb(r 17|Z(x))]** *by lines (I) to (V)*

(VII) **~{x B_c (17 Gen r) → Bew_c[Sb(r 17|Z(x))]** *expression* (15)

(VIII) **Bew_c[Sb(r 17|Z(x))] & ~Bew_c[Sb(r 17|Z(x))]** *contradiction – by lines (VII) and (VIII)*

Similarly, since from expression (16) expression (15) can be derived — and *vice versa* — and since expression (15) has already been proven inconsistent, expression (16) must be inconsistent too.

GÖDEL'S CONCLUSION AND ITS CONCLUSIVE REFUTATION

(Lxxxii) Gödel now says (and this is the culminating error in his so-called "proof"):

> Hence:
>
> 1. **17 Gen r** is not **c**-PROVABLE. For if that were so, there would (according to 6.1) be an n such that **n B_c (17 Gen r)**. By (16) it would therefore be the case that:
>
> **Bew_c[Neg Sb(r 17|Z(n))]**
>
> while–on the other hand–from the **c**-PROVABILITY of **17 Gen r** there follows also that of **Sb(r 17|Z(n))**. **c** would therefore be inconsistent (and, *a fortiori*, ω-inconsistent).
>
> 2. **Neg(17 Gen r)** is not **c**-PROVABLE. Proof: As shown above, **17 Gen r** is not **c**-PROVABLE, i.e. (according to 6.1) the following holds: **(n) ~[n B_c(17 Gen r)]**. Whence it follows, by (15), that **(n) Bew_c[Sb(r 17|Z(n))]**, which together with **Bew_c[Neg(17 Gen r)]** would conflict with the ω-consistency of **c**.
>
> **Neg(17 Gen r)** is therefore undecidable in **c**, so that Proposition VI is proved.

It should now be abundantly apparent, from all that has been pointed out heretofore by us, that Gödel's Theorem has more holes in it than a Swiss cheese, due to the fact that both the FORMULAE **17 Gen r** and **Sb(r 17|Z(n))** — and indeed the vast majority of Gödel's definitions 1. to 46., as well as his expressions (8.1) to (16) — are syncretic, and therefore cannot be part of a sound metamathematical proof. We shall now examine just a few additional holes in Gödel's argument hereunder, so as to demonstrate conclusively that Gödel's Theorem cannot possibly be proved in a valid manner.

First of all — and as we already mentioned in PART 1 of our critique — we note that Gödel's final argument is of the *reductio ad impossibile* form, whereby Gödel tries to show that assuming that the proposition "**17 Gen r** is **c**-PROVABLE" results in a contradiction.

However, the *reductio ad impossibile* argument is flawed, by the following consideration:

In Gödel's natural language argument No. 1. above, he makes use of the consequent of expression (16), namely

$$\mathbf{Bew_c[Neg\ Sb(r\ 17|Z(n))]}$$

… to attempt to prove that, in his words, "**17 Gen r** is not **c**-PROVABLE" — or symbolically, that **~**$\mathbf{Bew_c}$**(17 Gen r)** must hold.

However, if **~**$\mathbf{Bew_c}$**(17 Gen r)** holds, then there can be no proof in **c** of **17 Gen r** … or in other words, the antecedent of expression (16), namely **x** $\mathbf{B_c}$ **(17 Gen r)**, must be false (or perhaps we should say — to express it more accurately — incorrect).

And if the antecedent of expression (16), namely **x** $\mathbf{B_c}$ **(17 Gen r)**, is false (or incorrect), so must the consequent of expression (16), namely $\mathbf{Bew_c}$**[Neg Sb(r 17|Z(n))]**, be.[94]

In that case, Gödel must have used a false (or incorrect) premise — namely the consequent of expression (16), *i.e.*, $\mathbf{Bew_c}$**[Neg Sb(r 17|Z(n))]** — in his *reductio ad impossibile* argument to attempt to prove that **17 Gen r** is not **c**-PROVABLE.

And using a false (or incorrect) premise in the *reductio ad impossibile* argument renders the argument flawed, and therefore incorrect.[95]

---

[94] Although it is true that just because the antecedent of an implication is false, it does not means that its consequent is false too, we have to look at this case in the context of **Proposition V**. *If* **Proposition V** is provable (note however that Gödel does not give a *rigorous* proof of it), then expressions (15) and (16) are derived from expressions (3) and (4). Now if, as Gödel claims, the antecedent of expression (15) is true, that is to say, **17 Gen r** is *not* **c**-PROVABLE, then the consequent of expression (15), namely $\mathbf{Bew_c}$**[Sb(r 17|Z(n))]**, *must* be true, and cannot be false: because anything implied by a true proposition is true (*vide* Russell's and Whitehead's remarks in their Introduction to *Principia Mathematica*). And if the consequent of expression (15) is true, then the consequent of expression (16), namely $\mathbf{Bew_c}$**[Neg Sb(r 17|Z(n))]**, *cannot* be true, because if it were, then a proposition and its NEGATION would *both* be provable in the class **c** of ω-consistent FORMULAE, thereby rendering the class **c** inconsistent (and *a fortiori*, ω-inconsistent).

[95] Obviously one cannot properly use a false premise to "prove" that *another* premise is also false. Nor can one properly use only one *part* of a conditional — the antecedent alone or the consequent alone — to prove another formula, for it can be the case that the conditional itself is true (or correct) even though neither its antecedent nor its consequence is true (or correct). This can be illustrated by the following example. Suppose a chap says to his girl-friend: "If I win the lottery, I promise to buy you a diamond necklace." But even though he is quite sincere in saying so, he does *not* win the lottery; and so his girl-friend does not get her diamond necklace. She cannot, however, accuse him of lying, taking the second part of his promise *alone* as his entire promise, and use the fact that he did not buy her a diamond necklace as "evidence" of breach of contract in a lawsuit. He did not say "I promise to buy you a diamond necklace"; he said "*If* I win the lottery, I promise to buy you a diamond necklace". His promise in its entirety was *conditional*. Judge Judy would throw the girl-friend's case right out of court — and rightly so. Similarly, Gödel cannot validly take $\mathbf{Bew_c}$**[Neg Sb(r 17|Z(n))]** *alone* as true (or correct). Expression (16), being a conditional, is like the fellow's promise to his girl-friend. No single *part* of it, taken *separately* from the rest, can be validly taken as true (or correct) unless it has been *proven* so by a separate argument. If as Gödel claims in his natural language argument No. 1, **17 Gen r** *is* indeed not **c**-PROVABLE, then the antecedent of expression (16) must be *false* (or incorrect), and as a result, so must the consequent of expression (16) be; and being a false premise, $\mathbf{Bew_c}$**[Neg Sb(r 17|Z(n))]** cannot be used *by itself* in an argument to prove something *else* false (or incorrect), such as the hypothesis that **17 Gen r** is **c**-PROVABLE — which is what Gödel tries to do. Judge Judy would throw Gödel's case out as well … and once again, quite rightly so.

In which case, **17 Gen r** *cannot* be non-c-PROVABLE. Or in other words, **~Bew$_c$(17 Gen r)** cannot hold.

This can also be seen from the following (alternative) argument:

Assume that Gödel's *reductio ad impossibile* argument for 1. above is *not* flawed.

Now in Gödel's *reductio ad impossibile* argument No. 1. above, he assumes that the proposition he actually desires to prove — that is to say, the statement "**17 Gen r** is not c-PROVABLE", or **~Bew$_c$(17 Gen r)** — would *not* result in a contradiction. This assumption, however, must be erroneous.

For if Gödel *were* indeed right, and **17 Gen r** *were* in fact non-**c**-PROVABLE, then there could — according to expression (6.1) — *not* be any **n** in the class **c** of ω-consistent FORMULAE such that **n B$_c$ (17 Gen r)**. By (15) it should therefore be the case that:

Bew$_c$[Sb(r 17|Z(n))]

… which however is also impossible, since if there is no **n** in **c**, there can also be no **Z(n)** in **c** — and as a consequence, in the CLASS-SIGN **r**, the FREE VARIABLE **17** cannot be substituted by a non-existent **Z(n)**. Consequently there can be no **c-**PROVABLE FORMULA which arises as a result.

In other words (and to express the above argument in greater detail):

If **17 Gen r** *were* non-**c**-PROVABLE, then in the class **c** of ω-consistent FORMULAE, there can exist no natural number **n** such that **n** corresponds to a PROOF in **c** of **17 Gen r**.

And thus no **Z(n)** — *i.e.*, no NUMBER-SIGN for such a natural number **n** — can possibly exist in the class **c** of ω-consistent FORMULAE either.

And consequently, if one is restricted to the class **c** of ω-consistent FORMULAE**,** the operation of substituting the NUMBER-SIGN **Z(n)** for the FREE VARIABLE **17** in the CLASS-SIGN **r** cannot be carried out in **c** … because, of course, **Z(n)** cannot even exist in **c**: and, naturally, no FORMULA can possibly result in **c** as a consequence of an operation that cannot be carried out.

And since there is no way to PROVE a non-existent FORMULA, it cannot be PROVEN in the class **c** of ω-consistent FORMULAE.

Thus if **17 Gen r** *were* non-**c**-PROVABLE, it would never be possible to PROVE **Sb(r 17|Z(n))** in the class **c** of ω-consistent FORMULAE — or for that matter, anywhere else either: for the FORMULA **Sb(r 17|Z(n))** would not even belong to **c**.

This however contradicts expression (15), by which, if **17 Gen r** *were* non-**c**-PROVABLE, **Bew$_c$[Sb(r 17|Z(n))]** *should* hold: *i.e.*, that **Sb(r 17|Z(n))** *should* be **c**-PROVABLE.

Or expressing the argument in symbolic form:

If it is assumed that Gödel's first conclusion — expressed in line *(I)* below — is correct, the contradictions in the last two lines below, namely lines *(VIII)* and *(IX)*, must hold:

(I) Assume that **~[$\mathbf{Bew_c}$(17 Gen r)]** holds *Gödel's first "proven" conclusion*

(II) Then **(n) ~[n $\mathbf{B_c}$ (17 Gen r)]}** holds *by Gödel's Definition* 46. *and his expression* (6.1)

(III) **(n) ~[n $\mathbf{B_c}$ (17 Gen r)]} → n ∉ c** *by line (II)*

(IV) **n ∉ c → Z(n) ∉ c → [Sb(r 17|Z(n))] ∉ c**
by **n ∉ c → f(n) ∉ c**, *when* **f(n)** *denotes any function of* **n**

(V) **(n) ~[n $\mathbf{B_c}$ (17 Gen r)]} → $\mathbf{Bew_c}$[Sb(r 17|Z(n))]** *by Gödel's expression* (15)

(VI) Therefore **$\mathbf{Bew_c}$[Sb(r 17|Z(n))]** also holds. *by lines (II) and (V)*

(VII) But **$\mathbf{Bew_c}$[Sb(r 17|Z(n))]** cannot hold
because by line (IV), **Sb(r 17|Z(n))** *does not belong to* **c**

(VIII) Therefore **$\mathbf{Bew_c}$[Sb(r 17|Z(n))]** both holds and does not hold
contradiction – by lines(VI) and (VII)

(IX) Therefore **~[$\mathbf{Bew_c}$(17 Gen r)]** cannot hold, contradicting the Assumption in line *(I)*.
by line (VIII) being a contradiction

In other words, *if* we assume that Gödel's own argument *is* correct, then it follows as a logical consequence thereof that **17 Gen r** cannot be *non*-**c**-PROVABLE either.

From the above, we see that:

1. **$\mathbf{Bew_c}$(17 Gen r)** cannot hold *by Gödel's argument* 1.[96]

2. **$\mathbf{Bew_c}$[Neg (17 Gen r)]** cannot hold *by Gödel's argument* 2.

3. **~[$\mathbf{Bew_c}$(17 Gen r)]** cannot hold *by our argument above.*

Thus *neither* **$\mathbf{Bew_c}$(17 Gen r)** *nor* **~[$\mathbf{Bew_c}$(17 Gen r)]** can hold.

And this leaves only **~{$\mathbf{Bew_c}$[Neg(17 Gen r)]}** to be considered. Now if we assume that Gödel's expression (13) holds, then since by expression (13),

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[96] Of course as we showed above, this argument cannot hold up to scrutiny; but what we are demonstrating here is that even *if* we assume that Gödel has actually proved his Theorem, then it would mean that *none* of the following four — **$\mathbf{Bew_c}$(17 Gen r)**, **$\mathbf{Bew_c}$(Neg 17 Gen r)**, ~**$\mathbf{Bew_c}$(17 Gen r)** *nor* **~$\mathbf{Bew_c}$(Neg 17 Gen r)** — can hold: and that can only be the case if **17 Gen r** does not even belong to **c**.

$$\mathbf{17\ Gen\ r = Sb(p\ 19|Z(p)),}$$

… it follows that

$$\mathbf{Neg(17\ Gen\ r) = Neg\ Sb(p\ 19|Z(p))].}$$

And thus

$$\mathbf{\sim\{Bew_c[Neg(17\ Gen\ r)]\} = \sim\{Bew_c[Neg\ Sb(p\ 19|Z(p))]\}}.$$

But by expressions (3) and (4) of **Proposition V**, there can be *no* relation with (or for, or regarding) any natural number **p** that can possibly imply a FORMULA like

$$\mathbf{\sim\{Bew_c[Neg\ Sb(p\ 19|Z(p))]\},}$$

… since whether a relation **R** *holds* or does *not hold* with or between an **n**-tuple of natural numbers designated by the series $\mathbf{(x_1, \ldots, x_n)}$, the implied FORMULA, whether that FORMULA be

$$\mathbf{Bew\{Sb[r\ (u_1 \ldots u_n)|(Z(x_1) \ldots Z(x_n)]\}} \qquad \textit{by expression (3)}$$

… or

$$\mathbf{Bew\{Neg\ Sb[r\ (u_1 \ldots u_n)|(Z(x_1) \ldots Z(x_n)]\}} \qquad \textit{by expression (4)}$$

… is *always* of the "**Bew-**" or PROVABLE variety!

In other words, according to **Proposition V**, if in the class of ω-consistent FORMULAE **c**, there is a relation **P** — *any* relation **P** — that holds for a natural number **p** — *any* natural number **p** — then by expression (3) of **Proposition V**, in the class of ω-consistent FORMULAE **c**,

$$\mathbf{P(p) \rightarrow Bew_c[Sb(p\ u|Z(p))]}$$

— where **p** is a one-place RELATION-SIGN (*i.e.*, a CLASS-SIGN) and **u** is the FREE VARIABLE of it — must hold in the class of ω-consistent FORMULAE **c** also.

And likewise, if the relation **P** does *not* hold in the class of ω-consistent FORMULAE **c** for a natural number **p**; then by expression (4) of **Proposition V**, in the class of ω-consistent FORMULAE **c**,

$$\mathbf{\sim P(p) \rightarrow Bew_c[Neg\ Sb(p\ u|Z(p))]}$$

— where **p** is once again the same one-place RELATION-SIGN (*i.e.*, a CLASS-SIGN) and **u** is the FREE VARIABLE of it — must hold in the class of ω-consistent FORMULAE **c**.

And therefore, if $\mathbf{Bew_c[Neg\ Sb(p\ u|Z(p))]}$ holds in the class of ω-consistent FORMULAE **c** — and this *must* be the case, since it is always possible for *some* relation **P** *not* to hold with respect to a natural number **p**! — then there cannot be *any* FORMULA such as

$$\sim\{\mathbf{Bew_c[Neg\ Sb(p\ 19|Z(p))]}\}$$

… in the class of ω-consistent FORMULAE **c:** for that would mean that a FORMULA and its **negation**, namely:

$$\mathbf{Bew_c[Neg\ Sb(p\ u|Z(p))]}$$

… and

$$\sim\{\mathbf{Bew_c[Neg\ Sb(p\ 19|Z(p))]}\}$$ [97]

… would *both* hold in the in the class of ω-consistent FORMULAE **c**, thereby conflicting with its consistency (and *a fortiori*, with its ω-consistency as well.)

In other words, if **Proposition V** is assumed to be **c**-PROVABLE, then

$$\sim\{\mathbf{Bew_c[Neg\ Sb(p\ 19|Z(p))]}\}$$

… cannot possibly belong to **c**.

And if

$$\sim\{\mathbf{Bew_c[Neg\ Sb(p\ 19|Z(p))]}\}$$

… cannot belong to **c**, neither can

$$\sim\{\mathbf{Bew_c[Neg(17\ Gen\ r)]}\}$$

… with which, by the implications of expression (13), it must be identical (as a result of the equality-signs throughout that expression).

Thus we have also shown that:

4. $\sim\{\mathbf{Bew_c[Neg(17\ Gen\ r)]}\}$ cannot hold *by our argument here above.*

And thus, *if* Gödel's own argument *is* in fact correct, then *none* of the following four FORMULAE, namely:

1. $\mathbf{Bew_c(17\ Gen\ r)]}\}$
2. $\mathbf{Bew_c[Neg(17\ Gen\ r)]}\}$
3. $\sim\{\mathbf{Bew_c(17\ Gen\ r)]}\}$ and

[97] Again, it is to be noted that **p**, being a CLASS-SIGN, can have only *one* FREE VARIABLE in it; and thus the FREE VARIABLE **u** in the FORMULA $\mathbf{Bew_c[Neg\ Sb(p\ u|Z(p))]}$ must be identical to the FREE VARIABLE **19** in the FORMULA $\sim\{\mathbf{Bew_c[Neg\ Sb(p\ 19|Z(p))]}\}$.

4. **~{Bew_c[Neg(17 Gen r)]}**

… can hold.

And this is only possible if **17 Gen r** is *does not belong at all to* **c**: *i.e.*, if **17 Gen r** is not what Gödel calls a "meaningful formula" — or what we, in our days, would call a "well-formed formula" — belonging to what Gödel calls "the class **c** of ω-consistent FORMULAE."

And in that case, of course, it cannot be undecidable *in* **c**. (Nor, of course, can it be *decidable* in **c** … since the question of its undecidability — or even of its decidability — in **c** cannot arise, because it does not even *belong* to **c**.)

Additionally, since by expression (6.1),

Bew_c(17 Gen r) ≡ (∃x) x B_c (17 Gen r)

… and

~[Bew_c(17 Gen r)] ≡ (x) ~[x B_c (17 Gen r)],

… if neither **Bew_c(17 Gen r)** nor **~[Bew_c(17 Gen r)]** can hold, then neither **(∃x) x B_c (17 Gen r)** nor **(x) ~[x B_c (17 Gen r)]** can hold either — that is, the "natural number" **x** can neither exist nor *not* exist in **c**. This is utterly absurd … *unless* of course **x** is not in fact a natural number at all! (But we knew that all along, didn't we.)

PART 2-C

THE REST OF GÖDEL'S 1931 PAPER

(Lxxxiv) After this point the rest of our critique becomes essentially superfluous; but for the sake of being thorough, we shall continue our critique to the end of Gödel's 1931 Paper.

Gödel says:

> One can easily convince oneself that the above proof is constructive, i.e. that the following is demonstrated in an intuitionistically unobjectionable way: Given any recursively defined class **c** of FORMULAE: If then a formal decision (in **c**) be given for the (effectively demonstrable) PROPOSITIONAL FORMULA **17 Gen r**, we can effectively state:
>
> 1. A PROOF for **Neg(17 Gen r)**.
>
> 2. For any given **n**, a PROOF for **Sb(r 17|Z(n))**.
>
> I.e. a formal decision of **17 Gen r** would lead to the effective demonstrability of an ω-inconsistency.

Regarding Gödel's observation that his proof is constructive — *i.e.*, can be obtained using Constructivist (or Intuitionistic) logic exclusively — there are two objections.

In the first place, as we saw, Gödel-numbers are not a subset of natural numbers, since Gödel-numbers belong to the metalanguage **G** while natural numbers belong to the object-language **P**; and thus Gödel cannot validly obtain his definitions 1. to 46. using them, whether he uses Intuitionistic logic or classical — or for that matter, any other.

But even if it were granted that he could do so, and "prove" his Theorem without recourse to classical symbolic logic (*i.e.*, without assuming the law of the excluded middle), it still would not imply that the law of the excluded middle could not be used to *refute* him. As we noted earlier, if such a restriction were assumed, no classicist could accept Gödel's Theorem as valid while at the same time accepting classical symbolic logic as valid as well. Either the one or the other would have to be rejected by all classicists. The only people who could then accept Gödel's Theorem would have to be Intuitionists, Constructivists and/or Finitists.

And even they would not be able to overcome the first (and more important) objection, namely that Gödel-numbers cannot be considered to be a subset of natural numbers.

Moreover — as noted in our critique — when **n** does not even belong to **c**, **Z(n)** cannot belong to **c** either … and so, of course, neither can **Sb(r 17|Z(n))**; and as a result, a PROOF in **c** for **Sb(r 17|Z(n))** cannot be stated, whether using Intuitionistic logic or classical.

Or at best, when **Z(n)** does not belong to **c**, **Sb(r 17|Z(n)) ≡ r**, and thus a PROOF for **r** can, perhaps, be stated in **c** — if **r** is considered to be a PROPOSITIONAL FORMULA and not a CLASS-SIGN. But in that case, of course, **17** cannot exist as a FREE VARIABLE either: for there can be no FREE VARIABLE in a PROPOSITIONAL FORMULA, which by Gödel's own definition contains *no* FREE VARIABLES.

But if a PROOF for **r** — when **r** denotes a PROPOSITIONAL FORMULA and not a CLASS-SIGN — can be stated in **c**, it must be possible to state therein a PROOF for **17 Gen r** also: for it can be argued that if **r** is a PROPOSITIONAL FORMULA and as a consequence **17** does not exist in **c** as a FREE VARIABLE, **17 Gen r ≡ r**.

Thus one cannot give a formal decision in **c** — as Gödel does — that **17 Gen r** is *not* **c**-PROVABLE. The only formal decision that can be given under these circumstances for **17 Gen r** is that it *is* **c**-PROVABLE.

And if **17 Gen r** *is* **c**-PROVABLE, there cannot be a PROOF for **Neg(17 Gen r)**.

Consequently, a formal decision of **17 Gen r** would *not* lead to the effective demonstrability of an ω-inconsistency.

(Lxxxv) Then Gödel says:

> We shall call a relation (class) of natural numbers **R(x_1 … x_n) calculable** [*entscheidungs-definit*],[98] if there is an **n**-place RELATION-SIGN **r** such that (3) and (4) hold (cf. Proposition V). In particular, therefore, by Proposition V, every recursive relation is calculable. Similarly, a RELATION-SIGN will be called calculable, if it be assigned in this manner to a calculable relation. It is, then, sufficient for the existence of undecidable propositions, to assume of the class **c** that it is ω-consistent and calculable.

This of course is true only if **Proposition V** can actually be proved in the way Gödel tacitly intends it to be proved: that is, that the consequents of expressions (3) and (4) *both* hold.

But, as we have seen — and as we shall see below as well — that would entail a contradiction.

And of course, if the consequents of expressions (3) and (4) do *not* both hold, neither do those of expressions (9) and (10) which are derived from expressions (3) and (4); and as a result, neither do the consequents of expressions (15) and (16), which are derived from those of expressions (9) and (10).

But that cannot be, and still allow Gödel to "prove" his Theorem: for it will be noted that in the two *reductio ad impossibile* arguments he makes in natural language to attempt to demonstrate that neither **17 Gen r** nor **Neg(17 Gen r)** are **c**-PROVABLE, he makes use of *both*, the con-

[98] In *Kurt Gödel: Collected Works*, the German word *entscheidungsdefinit* is translated as "decidable" .

sequent of expression (16) — in his natural language argument No. 1. — *and* the consequent of expression (15) — in his natural language argument No. 2.

(Lxxxvi) Gödel continues:

> For the property of being calculable carries over from **c** to **x** $\mathbf{B_c}$ **y** (cf. (5), (6)) and to **Q(x,y)** (cf. 8.1), and only these are applied in the above proof.

This of course is not true at all: what is *actually* applied in the so-called "proof" is the definition of **Q(x,y)** as being identical to **~{x** $\mathbf{B_c}$ **[Sb(y 19|z(y)]**, and the further implied assertion that when **Q(x,y)** is so defined, it is an instance of the antecedent of expression (3) … which together result in a self-contradiction, and thus cannot be part of a consistent system of logic.

And moreover, **~{x** $\mathbf{B_c}$ **[Sb(y 19|z(y)]** by *itself* — even disregarding **Q(x,y)** — is inconsistent with expression (3) of **Proposition V**; and thus if expression (3) is assumed to belong to **c**, then the FORMULA **~{x** $\mathbf{B_c}$ **[Sb(y 19|z(y)]** cannot belong to **c**.

(Lxxxvii) Gödel then says:

> The undecidable proposition has in this case the form **v Gen r**, where **r** is a calculable CLASS-SIGN (it is in fact enough that **c** should be calculable in the system extended by adding **c**).

Parenthetically, if the word *entscheidungsdefinit* has been translated accurately as "calculable", the question should surely arise: when calculated, what does the CLASS-SIGN **r** calculate *to?*

For as we noted in *(Lxxviii)* above, the way Gödel has written his Theorem, **r** could represent *any* CLASS-SIGN in the class **c** of ω-consistent FORMULAE, and **17** could represent *any* FREE VARIABLE in the class **c** of ω-consistent FORMULAE.

But if so, what Gödel would have to be asserting is, that there is no PROOF in the class **c** of ω-consistent FORMULAE for the generalisation of *any* CLASS-SIGN in **c** by means of *any* FREE VARIABLE in **c**.

Or in other words, the term '**17 Gen r**' would not correspond to a *specific* PROPOSITIONAL FORMULA of the form **v Gen a**, but to *any* PROPOSITIONAL FORMULA of the form **v Gen a**.

And if, as Gödel claims, **17 Gen r** is not **c**-PROVABLE and neither is its NEGATION **c**-PROVABLE, then it should follow that *no* PROPOSITIONAL FORMULA of the type **v Gen a** is **c**-PROVABLE, and neither is the NEGATION of any PROPOSITIONAL FORMULA of the type **v Gen a c**-PROVABLE.

But this is obviously absurd … *unless* of course no PROPOSITIONAL FORMULA of the form **v Gen a** even *belongs* to the class **c** of ω-consistent FORMULAE!

It is also to be noted that Gödel does not actually *spell out* what the PROPOSITIONAL FORMULA **17 Gen r** should work out to in the system **P** (or any other consistent system), so as to allow his critics to thoroughly examine it and see whether it really *is* decidable or not — and also to check whether it really is a well-formed FORMULA capable of belonging to the system **P**, or not.

For if one restricts oneself to a *particular* class **c** of ω-consistent FORMULAE — such as for example the system **P**, or the axiom system for set theory of Zermelo-Fraenkel extended by John von Neumann — *if* Gödel is correct, it should be possible to actually *work out* or *enunciate* the PROPOSITIONAL FORMULA **17 Gen r** — so that we can say just which FREE VARIABLE is represented by the symbol **17**, and which CLASS-SIGN by the symbol **r** — and thereby allow the FORMULA **17 Gen r** to be tested. That is to say, it should be possible to *check* whether **17 Gen r** really *is* undecidable in that class, by enunciating what **17 Gen r** states when interpreted as to content. or when expressed in the exact notation of the system **P** or the axiom system for set theory of Zermelo-Fraenkel extended by John von Neumann (as the case may be).

As Gödel indicates in ***Section 1*** of his Paper, his propositional formula **[R(q); q]** "asserts its own unprovability": *i.e.*, it states, when interpreted as to its content, that **[R(q); q]** is unprovable. But it is not permissible in a metamathematical proof for a proposition to talk *about* itself, for that would mean that it must belong to both the object-language *and* the metalanguage, which in turn would contravene the use / mention rules for establishing a sound proof in a metalanguage. Nevertheless, since Gödel lays "no claim to exactness" in ***Section 1*** of his Paper, this oversight may be excused therein.

However, such an oversight cannot be excused in ***Section 2*** of Gödel's Paper, which purports to be a "*rigorous* development of the proof" outlined in ***Section 1***.

And yet neither Gödel himself, nor — to our knowledge — anyone else, has ever actually *enunciated* or *worked out* the PROPOSITIONAL FORMULA **17 Gen r**: that is to say, no one has actually expressed it in the exact notation of the system **P**, or in the notation of the axiom system for set theory of Zermelo-Fraenkel extended by John von Neumann: eschewing *all* of Gödel's own recondite notation as given in his 1931 Paper. For that matter, no one has even expressed it in natural language in such a manner that it stands up to criticism.

PUBLIC CHALLENGE

Indeed if **17 Gen r** *is* in fact undecidable *in* the system **P**, then it should be a FORMULA *of* the system **P** — and in which case, by the process of "Gödelisation" there ought to be a Gödel-number corresponding to it. And likewise, there ought also to be a Gödel-number corresponding to its NEGATION, namely the FORMULA **Neg(17 Gen r)**.

If so, however, it should be possible to state precisely, in digits, just *what* these two Gödel-numbers are.[99] And yet not only has Gödel failed to state them, but no one else, at least to our knowledge, has done so either.

And this, we contend, is because it just *cannot be done!*

Thus to any supporters of Gödel's Theorem we propose the following public challenge:

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[99] Once these Gödel-numbers are stated, of course, it should be possible, using these two Gödel-numbers and the reverse of the process of "Gödelisation", to reconstruct the exact FORMULAE of the system **P** to which these two Gödel-numbers correspond.

In a particular class **c** of ω-consistent FORMULAE — such as for example the system **P**, or the axiom system for set theory propounded by Zermelo and Fraenkel, and extended by John von Neumann — let our critics actually *work out* the PROPOSITIONAL FORMULA **17 Gen r** — *i.e.*, express it in the *exact notation* of the system **P** or of the axiom system for set theory propounded by Zermelo and Fraenkel, and extended by John von Neumann (as the case may be), or even in natural language, eschewing *all* of Gödel's own made-up notation — and show that it can be done. And let them also show that it really *is* undecidable in the system **P** or the axiom system for set theory of Zermelo-Fraenkel extended by John von Neumann (as the case may be), and that it *is* a well-formed formula of that system.

Why, we could even simplify the challenge: let our critics restrict themselves to just the system **P**, and let them merely state precisely, in digits, the exact Gödel-number of the purported undecidable FORMULA, if they can! Just *one* Gödel-number, no more — that alone should suffice to test whether an undecidable FORMULA exists at all in the system **P**, or not.

If they cannot do this, then, we contend, there *is no proof* of the existence of any undecidable PROPOSITIONAL FORMULA therein: and as a result, we can safely claim that at the very least, Gödel cannot have actually *proved* his Theorem. And that, we contend, should be sufficient to silence all supporters of Gödel's "proof".

(Lxxxviii) Gödel goes on thus:

> If, instead of ω-consistency, mere consistency as such is assumed for **c**, then there follows, indeed, not the existence of an undecidable proposition, but rather the existence of a property **(r)** for which it is possible neither to provide a counter-example nor to prove that it holds for all numbers. For, in proving that **17 Gen r** is not **c**-provable, only the consistency of **c** is employed (cf. [189]) and from **~Bew$_c$(17 Gen r)** it follows, according to (15), that for every number **x**, **Sb(r 17|z(x))** is **c**-PROVABLE, and hence that **Neg [Sb(r 17|Z(x))]** is not **c**-PROVABLE for any number.

Now this is utter nonsense, since in order to "prove" that **17 Gen r** is not **c**-PROVABLE, Gödel has already made use of the consequent of expression (16), namely

Bew$_c$[Neg Sb(r 17|Z(x))]

Now he also intends to make use of the consequent of expression (15) — namely

Bew$_c$[Sb(r 17|Z(x))]

… to "prove" that **Neg [Sb(r 17|Z(x))]** is not **c**-PROVABLE for any number: *i.e.*, that

~{Bew$_c$[Neg Sb(r 17|Z(x))]}

… must hold.

But since there is a clear and obvious contradiction between the consequent of expression (15) and the consequent of expression (16), is it any surprise that there is a contradiction between the consequent of expression (16) and the term Gödel derives from the consequent of expression (15)? Of course it isn't, since he has already *accepted* the ludicrous notion that the consequent of expression (15) and the consequent of expression (16) — which contradict each other — can *both* be used in a *single* argument![100]

This, again, is obviously an inconsistency; and deriving another inconsistency therefrom merely compounds the original error.

(Lxxxix) Gödel continues:

> By adding **Neg(17 Gen r)** to **c**, we obtain a consistent but not ω-consistent class of FORMULAE **c'**. **c'** is consistent, since otherwise **17 Gen r** would be **c**-PROVABLE. **c'** is not however ω-consistent, since in virtue of **~Bew_c(17 Gen r)** and (15) we have: **(x) Bew_cSb(r 17|Z(x))**, and so *a fortiori:* **(x) $Bew_{c'}$Sb(r 17|Z(x))**, and on the other hand, naturally: **$Bew_{c'}$[Neg(17 Gen r)]**.

And this too is nonsense, since as we pointed out above, in order to "prove" that **17 Gen r** is not **c**-PROVABLE, Gödel has already made use of the consequent of expression (16), namely

Bew_c[Neg Sb(r 17|Z(x))]

Now he intends to make use of the consequent of expression (15) — namely

Bew_c[Sb(r 17|Z(x))]

… to "prove" that both **$Bew_{c'}$Sb(r 17|Z(x))** and **$Bew_{c'}$[Neg(17 Gen r)]** hold simultaneously. Of course they do, since he has already accepted the inconsistency of the consequent of expression (15) and the consequent of expression (16) — which blatantly and shamelessly contradict each other — *both* capable of being used in a single argument.

But that is obviously an inconsistency; and deriving another inconsistency therefrom merely renders confusion worse confounded.

(xC) Gödel now says:

> A special case of Proposition VI is that in which the class **c** consists of a finite number of FORMULAE (with or without those derived therefrom by TYPE-LIFT). Every finite class **a** is naturally recursive. Let **a** be the largest number contained in **c**. Then in this case the following holds for **c**:
>
> $$x \in c \equiv (\exists m,n)[m \leq x \ \& \ n \leq a \ \& \ n \in a \ \& \ x = m \ Th \ n]$$
>
> **c** is therefore recursive. This allows one, for example, to conclude that even with the help of the axiom of choice (for all types), or the generalized continuum hypothesis, not all propositions are decidable, it being assumed that these hypotheses are ω-consistent.

[100] Note, moreover, that not only does Gödel *use* them both, but he also tacitly assumes them both to be *correct* as well.

Obviously the correctness of all this depends on Gödel having actually proved his **Proposition VI**. But he obviously hasn't, so there is no reason to conclude that what he says above is correct, either.

(xCi) Gödel continues:

> In the proof of Proposition VI the only properties of the system **P** employed were the following:
>
> 1. The class of axioms and the rules of inference (i.e. the relation "immediate consequence of") are recursively definable (as soon as the basic signs are replaced in any fashion by natural numbers).
>
> 2. Every recursive relation is definable in the system **P** (in the sense of Proposition V).

As we have seen in *(Lxviii)* above, however (*q.v.*), if the "basic signs are replaced in any fashion by natural numbers", and no distinction is made between the Gödel-numbers so derived and the natural numbers *of* the system **P** — *i.e.*, if Gödel-numbers are taken to be a sub-set of natural numbers — it renders the system **P** inconsistent,[101] by allowing self-contradictory expressions, such as expression (8.1), to be enunciated therein: and as a result, no conclusions derived therefrom can logically hold.

Moreover, as we have also seen in *(Lxxvii)* above (*q.v.*), the so-called recursive relations — such as the antecedents of Gödel's expressions (9) and (10) — are *not* definable in the system **P**.

Thus nothing that Gödel says above holds true at all.

(xCii) Gödel now says:

> Hence in every formal system that satisfies assumptions 1 and 2 and is ω-consistent, undecidable propositions exist of the form **(x) F(x)**, where **F** is a recursively defined property of natural numbers, and so too in every extension of such a system made by adding a recursively definable ω-consistent class of axioms.

But since as we have shown, no consistent (what to speak of "ω-consistent"!) formal system satisfies assumptions 1. and 2. above, no undecidable propositions of the form **(x) F(x)** — where **F** is a recursively defined property of natural numbers — exist at all; and thus, neither do they exist in any "extension of such a system made by adding a recursively definable ω-consistent class of axioms".

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[101] Take for instance the basic signs of the system **P** which represent the natural number **2**, namely ***ff*0**. By the procedure we have called "Gödelisation", these would end up as $\mathbf{2^3 \cdot 3^3 \cdot 5^1}$ or **8·27·5**, which happens to be **1,080**. Replacing ***ff*0** by **1,080** in the equation ***ff*0 = 2**, one gets the equation **1,080** = **2**. But in the system **P**, **1,080 ≠ 2**. So we get **(1,080** = **2)** & (**1,080 ≠ 2)**, which results in an inconsistency being demonstrated in the system **P**.

(xCiii) Gödel continues:

> As can be easily confirmed, the systems which satisfy assumptions 1 and 2 include the Zermelo-Fraenkel and the v. Neumann axiom systems of set theory, and also the axiom system of number theory which consists of the Peano axioms, the operation of recursive definition [according to schema (2)] and the logical rules.

As is easily seen now, however, neither the Zermelo-Fraenkel nor the von Neumann axiom systems of set theory, nor even the axiom system of number theory which consists of the Peano axioms, the operation of recursive definition [according to schema (2)] and the logical rules, satisfy assumptions 1. and 2.

(xCiv) Gödel finally says:

> Assumption 1 is in general satisfied by every system whose rules of inference are the usual ones and whose axioms (like those of **P**) are derived by substitution from a finite number of schemata.

This is quite untrue, since as soon as the basic signs of the system **P** are replaced by natural numbers by the process of "Gödelisation", the entire system is rendered syncretic, and thus incapable of being used to soundly establish a metamathematical proof. That is because, as we have just seen in *(xCi)* above (*q.v.*), if no distinction is made between object-language and metalanguage (which is what Gödel does), the system of reasoning as a whole — *i.e.*, object-language-plus-metalanguage — is rendered inconsistent.

And on the other hand, when a distinction *is* made between object-language symbols and metalanguage symbols, the vast majority of Gödel's definitions 1. to 46., as well as the majority of his expressions (3) to (16), cannot be asserted, for they contain both metalanguage symbols as well as object-language symbols, and are thereby rendered nonsensical or meaningless: as nonsensical as the statement "A ripe 'grape' is a five-letter word that tastes sweet."

(xCiv) In ***Section 3*** of his Paper, Gödel begins by saying:

> From Proposition VI we now obtain further consequences … (*etc.*).

And in ***Section 4*** of his Paper, he begins by saying:

> From the conclusions of ***Section 2*** there follows a remarkable result … (*etc.*).

However, since all he has to say in his ***Sections 3*** and ***4*** depends on the conclusions of his ***Section 2*** — and in particular of **Proposition VI** — being correctly proven, and since as we have seen above these conclusions *cannot* be considered to be correctly proven, nothing he has to say in ***Sections 3*** and ***4*** need be held as correctly proven either.

So here we rest our case.

SUMMARY

Summarising, Gödel's so called "Incompleteness Theorem" is not logically valid and cannot be logically valid because:

1. The procedure of "Gödelisation" — which essentially results in the unification of the mathematic and metamathematic language levels — is not a valid procedure for the establishment of a sound proof in metamathematics: since when such a procedure is used, a single symbol is allowed to represent two distinct concepts belonging to two different languages. If *no* distinction is made between object-language terms and metalanguage terms, those concepts can clash with one another, resulting in the possibility of deriving contradictions — thus rendering inconsistent the system of logic used for the argument. Whereas if a distinction *is* made between object-language terms and metalanguage terms, syncretism arises, allowing a single expression to include terms from both languages, which contravenes the rules soundly for establishing a metamathematical proof.

   Neither classical symbolic logic, nor — for that matter — any other logic, admits total and *absolute* contradictions as valid expressions, but rather as invalid expressions. Thus when one or more total and absolute contradictions can be derived using a particular process of reasoning, we conclude that the entire process of reasoning must be logically invalid, and that the conclusions derived therefrom cannot hold in *any* system of logic.

   As we have seen, if no distinction is made between object-language terms and metalanguage terms — which is the procedure Gödel follows — a contradiction arises between the implication of the definiendum of Gödel's expression (8.1) and the implication of the definiens of expression (8.1). The implication of the definiendum of expression (8.1) — namely, that a natural number **x** *must* exist in the class **c** of ω-consistent FORMULAE — totally and absolutely contradicts the implication of the definiens of expression (8.1), namely that a natural number **x** *cannot* exist in the class **c** of ω-consistent FORMULAE. As a result, if no distinction is made between object-language terms and metalanguage terms, expression (8.1) ends up being self-contradictory.

   And another such contradiction, as we have also seen, arises between expression (8.1) and Gödel's **Proposition V**. If **Proposition V** is PROVABLE in a class **c** of ω-consistent FORMULAE, then the definiens of expression (8.1), being a counter-example of the consequent of expression (3) of **Proposition V**, cannot exist in the same class **c** of ω-consistent FORMULAE.

   And since Gödel *needs* expression (8.1) to attempt to "prove" his Theorem — and indeed even to derive his so-called "undecidable" PROPOSITIONAL FORMULA **17 Gen r** — it means that he has used a FORMULA that does not exist in the class **c** of ω-consistent FORMULAE to derive the "undecidable" PROPOSITIONAL FORMULA **17 Gen r**: and thus **17 Gen r** itself cannot exist in the class **c** of ω-consistent FORMULAE either.

2. Gödel uses a *reductio ad impossibile* argument to attempt to prove his conclusion. But as we noted, his *reductio ad impossibile* argument for the first part of his natural-language conclusion, namely that "**17 Gen r** is not **c**-PROVABLE", is flawed: in that he makes use of a premise in it — namely the consequent of expression (16) — which, after the first part of the argument is completed, is seen to have been falsely assumed to have been correct, when in point of fact it could not have been so.

   Moreover, a *reductio ad impossibile* argument has a serious limitation: namely, it tacitly *assumes* that the denial of the proposition which generates a contradiction does *not* itself generate a contradiction. Gödel shows, using a *reductio ad impossibile* argument, that assuming that **17 Gen r** *is* **c**-PROVABLE generates a contradiction: but he does *not* examine whether the denial of it, namely assuming that **17 Gen r** is *not* **c**-PROVABLE, generates a contradiction also.

   We have therefore rectified this oversight on Gödel's part, and examined the proposition that **17 Gen r** is *not* **c**-PROVABLE: and we have found that assuming that **17 Gen r** is *not* **c**-PROVABLE *also* generates a contradiction: for by Gödel's expression (15), such an assumption requires the simultaneous existence and non-existence of a natural number **n** in the class **c** of ω-consistent FORMULAE.

   Thus using two *reductio ad impossibile* arguments, Gödel's expression **17 Gen r** is found to be not **c**-PROVABLE and also not *non*-**c**-PROVABLE.

   And of course, Gödel also proves, using yet another *reductio ad impossibile* argument, that **Neg(17 Gen r)** is not **c**-PROVABLE. And as we have shown in *(Lxxxii)*, it can also not be *non*-**c**-PROVABLE. Thus neither **17 Gen r** nor **Neg(17 Gen r)** is **c**-PROVABLE nor *non*-**c**-PROVABLE.

   But this is absolutely and categorically not possible if **17 Gen r** belongs to a *consistent* class of FORMULAE. It is *only* possible if **17 Gen r** does *not* belong to a consistent class of FORMULAE — and, *a fortiori*, to an ω-consistent class of FORMULAE.

   And thus Gödel's **Proposition VI** cannot hold, *i.e.*, no undecidable proposition can be found *in* what Gödel calls an "ω-consistent class **c** of FORMULAE".

As a consequence of the two salient points above, Gödel's so-called "Incompleteness Theorem" stands conclusively refuted.

Indeed, all that has been accomplished with Gödel's 1931 Paper is the generation of a great deal of publicity, due to the obscure and convoluted, yet intricate, manner in which it is written: for when a person reads it, he or she can hardly even begin to understand it, and as a result feels: "This stuff is so complicated, and has so many pages and pages of long and intricate formulae, that it *must* be correct: no reasonable person would otherwise have spent the time and energy to write it all out; and surely Gödel must have been a reasonable person, because otherwise he would never have been appointed a Professor at a prestigious institution like the *Institute for Advanced Study* at Princeton, nor could he have had a brilliant chap like Einstein for a close personal friend." Gödel's 1931 Paper, in short, is barely intelligible, and yet due to historical reasons, Gödel has acquired enormous prestige in the minds of a large number of people, including those who cannot fully understand his argument, but accept it because they think it *must* be true. This in turn creates a psychological and rhetorical effect on its readers, and a definite reluctance to take the time and make the effort to carefully examine the errors inherent in Gödel's Paper.

The problem has, in fact, become so acute that at present Gödel's Theorem is accepted virtually as dogma in all the world's universities, and trying to convince a professor of logic or mathematics — especially one with tenure — to admit to even the possibility that it *might* contain errors is almost like trying to convince the Ayatollah Khomeini that there might be errors in the Qr'an. Perhaps it only will be the next generation that will be sufficiently open-minded to refuse to become "Gödel-worshippers", and as a result, to examine Gödel's 1931 Paper critically for themselves, and eventually to notice the fact that the syncretism which arises due to the unification of the mathematic and metamathematic language levels is not a valid procedure for the sound establishment of a proof in metamathematics.

COMMENTS

Comments, if any, will be most welcome. The authors appreciate e-mail sent to them at their e-mail addresses, given on the Title Page. In forthcoming editions of our Critique we shall include some of the more interesting comments from our readers on this page and the following pages.